Set 1 [10]

1. Yu has 100 apples, and You have none. You take away 33 apples from Yu. How many apples do You now have?

   Answer: $\boxed{33}$ *(Proposed by: Alex Tsagaan)*

2. Compute the unique 3-digit integer, n, such that $\binom{n}{3}$ is divisible by 10000.

   Answer: $\boxed{626}$ *(Proposed by: Neil Dixit)*

3. Given an integer n, consider a set of points of each of its factors. You can travel from one factor, a, to another factor, b, if and only if b is a divisor of a and $b \neq a$. The cost of any such move is the value of b. For the value $n = 360$, find the cost of the highest-weight path from 360 to 1. For example, one possible path would be $360 \to 120 \to 24 \to 6 \to 1$, and summing up the costs gives $120+24+6+1 = 151$.

   Answer: $\boxed{336}$ *(Proposed by: Neil Dixit)*

Set 2 [11]

4. Let R denote the region above the parabola $y = x^2 + 7$ and below the line $y = 23$. Let S denote the region formed when you rotate R a full revolution ($360°$) about the origin in the xy plane. The area of S is $n\pi$, where n is an integer. Compute n.

   Answer: $\boxed{496}$ *(Proposed by: Neil Dixit)*

5. Take a sector of a circle of radius 14, and fold it so that it makes a cone with an open base (like an ice-cream cone). To that open base, attach a hemisphere. Given that this new shape's surface area is 156π, its volume can be expressed in the form $\pi(a\sqrt{b} + c)$, where a, b, and c are positive integers and c is square-free. Compute $a + b + c$.

   Answer: $\boxed{202}$ *(Proposed by: Neil Dixit)*

6. Mary has 10 pairs of gloves. Oh no! They got mixed up in her drawer! Two pairs are red, three pairs are green, three pairs are blue and two pairs are purple. Mary also counts a purple and a blue glove as a pair, since they look similar enough from far away. The chances Mary has a pair of gloves after randomly taking three gloves out (without replacement) can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

   Answer: $\boxed{34}$ *(Proposed by: Grace Liu)*

Set 3 [13]

7. Let $P(x)$ be a non-linear polynomial with positive integer coefficients such that $P(0), P(1), P(2), \ldots, P(10)$ are in increasing order, all positive, and all divisible by 10. Find the smallest possible value of $P(10)$.

   Answer: $\boxed{560}$ *(Proposed by: Neil Dixit and Alex Tsagaan)*

8. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD, BC = AD$, and $AB = 8$. A circle, ω, is inscribed within $ABCD$ (internally tangent to all 4 sides), and has center I. Given that the area of $\triangle BCI$ is 5 times the area of $\triangle ABI$, compute the area of trapezoid $ABCD$.

   Answer: $\boxed{960}$ *(Proposed by: Neil Dixit)*

9. Player 1 and Player 2 are playing a game against each other. They stop playing when one player has 2 more wins than the other player, and that player wins the match. Suppose that on odd-numbered games (1st game, 3rd game, 5th game etc.), Player 1 has a $\frac{4}{5}$ chance of winning, and on even-numbered games, Player 2 has a $\frac{4}{5}$ chance of winning. The expected number of games until someone wins is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: $\boxed{29}$ *(Proposed by: Aarush Rachakonda)*

Set 4 [14]

10. The digits of 3! do not end in a zero. The digits of 6! end in one zero. The digits of 12! end in two zeroes. For how many values of k from 0 to 999 does there exist n such that $n!$ ends in exactly k zeroes (the digit before those k zeroes must be nonzero)?

    Answer: $\boxed{801}$ *(Proposed by: Neil Dixit)*

11. x and y are independently chosen randomly from the interval $[0, \sqrt{3}]$. Let P denote the probability that if x and y are the two legs of a right triangle, the hypotenuse is greater than 2. Compute $\lfloor 1000P \rfloor$.

    Answer: $\boxed{73}$ *(Proposed by: Neil Dixit)*

12. Let $x_1, x_2, x_3, \ldots x_{67}$ be a set of 67 positive real numbers such that

    $$x_1 + x_2 + x_3 + \ldots x_{67} = 100.$$

    The maximum possible value of the expression

    $$\sqrt{x_1 - x_1x_2} + \sqrt{2x_2 - x_2x_3} + \sqrt{3x_3 - x_3x_4} + \ldots \sqrt{66x_{66} - x_{66}x_{67}} + \sqrt{67x_{67} - x_{67}x_1},$$

    given that it's real, can be expressed in the form $a\sqrt{b}$, where a and b are positive integers and b is square-free. Compute $a + b$.

    Answer: $\boxed{332}$ *(Proposed by: Neil Dixit)*

Set 5 [16]

13. A quadrilateral has side lengths $6, 7, 6, 9$ (not necessarily in that order). Then the maximum possible area of this quadrilateral can be expressed as $a\sqrt{b}$, where a, b are positive integers and b is squarefree. Compute $a + b$.

Answer: $\boxed{43}$ *(Proposed by: Neil Dixit and Alex Tsagaan)*

14. Let ABC be an acute triangle with orthocenter H. Let M be the midpoint of BC. Given that $\angle MAC + \angle BHC = 150°$ and $2HM = MC$, the value of $\sin(\angle ABC)$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$. Compute $a + b + c$.

Answer: $\boxed{26}$ *(Proposed by: Alex Tsagaan)*

15. Define functions f and g that take in an input from $S = \{1, 2, 3\}$ and output a value in S (which can be the same as the input). A function, f is called idempotent if it satisfies $f(f(n)) = f(n)$ for all possible n. Calculate the number of ordered pairs of idempotent functions (f, g) that satisfy $f(g(n)) = g(f(n))$ for all possible n.

Answer: $\boxed{58}$ *(Proposed by: Neil Dixit)*

Set 6 [17]

16. Let a, b, c be positive real numbers and let $k = abc$. Given that

$$(\log_k(a))^3 + (\log_k(b))^3 + (\log_k(c))^3 = \frac{2}{3} \text{ and}$$

$$(\log_a(k))(\log_b(k))(\log_c(k)) = 7776,$$

compute

$$\log_a(bc) + \log_b(ac) + \log_c(ab).$$

Answer: $\boxed{862}$ *(Proposed by: Neil Dixit)*

17. Consider a right triangle ABC with $\angle ABC = 90°$, and $AB = 8$. Given that the centroid of the triangle lies on its incircle, then the largest possible value of the side length BC can be expressed as $A + B\sqrt{C} + \sqrt{D\sqrt{E} + F}$, where A, B, C, D, E, F are all (not necessarily positive) integers such that the expression is in simplest form. Compute $A + B + C + D + E + F$.

Answer: $\boxed{50}$ *(Proposed by: Alex Tsagaan)*

18. Define $a^{-1} \pmod{b} \equiv x$ if x is the unique integer between 1 and $b - 1$ such that $ax \equiv 1 \pmod{b}$, and if no such integer exists, let this value be 0. (typically it's undefined). We define

$$f(n) = \sum_{k=1}^{n-1} \left(\left(k^2\right)^{-1} \pmod{n} \right).$$

Compute the remainder when $f(729)$ is divided by 1000.

Answer: $\boxed{904}$ *(Proposed by: Neil Dixit)*

Set 7 [19]

19. Let ABC be an acute triangle with $\angle A = 60^\circ$, $AB = 2$, and $AC = 3$. Let M be the midpoint of BC, and let BE and CF be altitudes that intersect at point H. Denote D as the reflection of A about point M. Let point P be the other intersection between circumcircles (BFM) and (CEM). Then $\tan \angle PDH$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that b is square-free and a, c are relatively prime. Compute $a + b + c$.

    Answer: $\boxed{47}$ *(Proposed by: Alex Tsagaan)*

20. Given an integer n, define a directed graph with all of its divisors as nodes. There exists an edge from one node, a, to another node, b, if and only if $\frac{a}{b}$ is a prime number. The weight of any such edge is the sum of the divisors of b. For the graph for $n = 720$, consider all possible paths from the node 720 to the node 1. One example path is the largest-weight path, which goes $720 \to 360 \to 180 \to 90 \to 45 \to 15 \to 5 \to 1$, and summing the sums of factors gives a weight of $1170 + 546 + 234 + 78 + 24 + 6 + 1 = 2059$. If I randomly choose one of these paths, its expected weight is equal to $\frac{m}{n}$, where m and n are relatively prime integers. Compute the remainder when $m + n$ is divided by 1000.

    Answer: $\boxed{617}$ *(Proposed by: Neil Dixit)*

21. Define a function $f(n)$ on the integers $n \geq 1$ such that

$$f(n) = \prod_{k=1}^{n} \left((k+1)^{k+1} - k^k\right).$$

    Compute the sum of all possible values of the last two digits of $f(n)$.

    Answer: $\boxed{860}$ *(Proposed by: Neil Dixit and Alex Tsagaan)*

Set 8 [20]

22. Neil starts at the origin in the xy plane. If Neil is at (x, y), he can move to $(x+1, y)$ or $(x, y+1)$. Neil wants to only go through points where the x and y coordinate are equal or relatively prime and wants to reach the point $(67, 67)$. How many of the $\binom{134}{67}$ paths can Neil take?

    Answer: $\boxed{1.964 \times 10^{20}}$ *(Proposed by: Neil Dixit)*

23. Consider the set of all possible triangles with its greatest side length less than 2025. If one of these triangles are chosen at random, what is the expected value of its area?

    Answer: $\boxed{237821}$ *(Proposed by: Neil Dixit)*

24. How many triangles with 3 integer sides all less than or equal to 2026 have at least one angle that, when rounded to the nearest degree, is 67°?

    Answer: $\boxed{2.02 \times 10^{7}}$ *(Proposed by: Neil Dixit)*