Individual Test (Division A)

Gunn Math Competition 2025

Instructions and Basic Format

- This round contains 10 short-answer questions to be solved in 60 minutes, by yourself. This means that you are not allowed to discuss this problem with anyone outside of your team until Lunch. Each problem is worth more points than all previous points, but by no significant amount, so be mindful of where you spend your time. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.
- **NO CALCULATORS (or abaci).** Protractors, rulers, and compasses are permitted. Do not cheat in any way. When caught, you will be blacklisted from the competition and will not be able to participate for the rest of the day.
- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2}+\frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π, e, $\sin 10°$, and $\ln 2$ in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

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Problems

1. Given that $x + \frac{1}{x} = 7$, what is $x^3 + \frac{1}{x^3}$ equal to?

2. Calculate the sum of the exponents in the prime factorization of 25!. For example, taking $6! = 720$, we have $720 = 2^4 \cdot 3^2 \cdot 5^1$, so the desired sum would be $4 + 2 + 1 = 7$.

3. Consider the maximum possible value of the following expression where x is a positive, real number:

$$\frac{4x+7}{x^2+3}.$$

   It can be expressed in the form $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers such that b is square-free and a and c are relatively prime. Compute $a + b + c$.

4. Let a, b, and c be the roots of $x^3 - 2x^2 + 3x + 4 = 0$. Let $x^3 + px^2 + qx + r = 0$ denote the polynomial with roots a^2, b^2, and c^2. Find $p + q + r$.

5. Consider two concentric circles of radii 1 and 2. Choose a point, P that is inside the larger circle but outside the smaller circle, and is a distance x from their centers. Draw a line segment starting at P that is tangent to the smaller circle, and ends on the outer circle. Given that the line segment's length is 3, compute x^2.

6. Satvik wants to reach the cafeteria. He starts at the origin and if he is at position (x, y), he can only move to positions $(x + 1, y + 1)$,$(x + 2, y)$, or $(x, y + 2)$. How many paths can Satvik take to reach the position $(7, 13)$, where he can eat lunch?

7. Let $ABCD$ be a cyclic quadrilateral such that $\overline{AB} = 5$, $\overline{BD} = 10$, $\overline{DA} = 8$, and $\overline{BC} = \overline{CD}$. Let P denote the intersection of segments BD and AC. Compute $\overline{BP}$.

8. Consider $\triangle ABC$ where $\overline{AB} = 29$, $\overline{BC} = 30$, and $\overline{CA} = 31$. Let D be a point inside $\triangle ABC$ such that $\angle DAC = \angle ACB$, and $\angle ADB = 90^{\circ} + \angle ACB$. Extend line AD until it intersects BC at point E. Compute $\frac{AD}{BE}$.

9. Compute the fourth-smallest prime factor of $2025^{109} - 1$.

10. Let S denote the infinite set of nonzero squares as follows:

$$S = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, ...\}.$$

   For any subset of S, let M denote the smallest value in that subset. Compute the variance of M given that each subset is equally likely to be selected.