Guts Round, Division B

Division B, 90 Minutes in Teams of 4, Sunday March 30th, 2025

Set 1 [10]

1. Farmer John has been in the cow business for decades, but he has now gotten bored of them. He now has a farm with some numbers of unicorns and peacocks. If the total number of mouths that the farmer needs to feed is 24 and the total number of legs is 74, how many peacocks does the farmer have?

   Answer: $\boxed{11}$ *(Proposed by: Grace Liu)*

2. Using a combination of pennies, nickels, dimes, and quarters, what is the largest total amount of money in dollars you can have without being able to create exactly one dollar using any combination of those coins?

   Answer: $\boxed{1.19}$ *(Proposed by: Grace Liu)*

3. Computer the remainder when 11^{73} is divided by 100.

   Answer: $\boxed{31}$ *(Proposed by: Neil)*

Set 2 [11]

4. Given a right triangle with side lengths a, b, and c and circumradius R, compute

$$\frac{a^2+b^2+c^2}{R^2}.$$

   Answer: $\boxed{8}$ *(Proposed by: Neil Dixit)*

5. Compute the sum of all positive integers that are equal to 700 times the sum of their digits.

   Answer: $\boxed{21000}$ *(Proposed by: Mary Yu)*

6. How many ordered pairs of positive integers (x, y) satisfy $xy + 3x + 7y = 79$?

   Answer: $\boxed{3}$ *(Proposed by: Neil Dixit)*

Set 3 [12]

7. Let α, β, γ, and δ be the roots of $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$. Compute

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} + \frac{1}{\delta+1}.$$

Answer: $\boxed{\frac{52}{61}}$ *(Proposed by: Neil Dixit)*

8. A rhombus of side length 7 has two diagonals, one of which is 7 times longer than the other. Compute the area of the rhombus.

Answer: $\boxed{\frac{343}{25}}$ *(Proposed by: Neil Dixit)*

9. Neil wants to arrange the integers from 1 to 2025 such that if the set of all integers divisible by 3 or 5 are in order. For example, the number 50 must come after 45 which must come after 42, but the number 46 can be placed anywhere. In how many ways can Neil accomplish this?

Answer: $\boxed{\frac{2025!}{945!}}$ *(Proposed by: Neil Dixit)*

Set 4 [13]

10. Let $f(x) = x^4 - px^3 + 54x^2 - qx + r$ have 4 real, positive roots. Compute the maximum possible value of $f(1)$.

Answer: $\boxed{16}$ *(Proposed by: Neil Dixit)*

11. Suppose we have two circles of radii 7 and 8 that are externally tangent at point P. There is another common tangent line to the circles at points Q and S respectively. Let R denote the intersection of the tangent through P with QS. Compute PR.

Answer: $\boxed{2\sqrt{14}}$ *(Proposed by: Neil Dixit)*

12. Calculate the number of positive integers less than or equal to 7! that are not divisible by any of 2, 3, 4, 5, 6, or 7.

Answer: $\boxed{1152}$ *(Proposed by: Neil Dixit)*

Set 5 [14]

13. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights a and b ($b < a$) below the apex. Given that the volume of the frustum equals its height, compute the minimum possible value of b.

    Answer: $\boxed{\sqrt{\frac{1}{\pi}}}$ *(Proposed by: Neil Dixit)*

14. The ζ (zeta) function is defined as

$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}.$$

    Compute

$$\sum_{k=1}^{\infty} (\zeta(2k) - 1)$$

    Answer: $\boxed{\frac{3}{4}}$ *(Proposed by: Neil Dixit)*

15. Alex chooses each of 2025 integers, $\{x_1, x_2, x_3, \ldots x_{2025}\}$ as a random integer from 1 to 2025 (with replacement). Compute the expected number of distinct x_i that Alex gets.

    Answer: $\boxed{2025(1 - (\frac{2024}{2025})^{2025})}$ *(Proposed by: Alex Tsagaan)*

Set 6 [16]

16. Suppose we have a list of positive numbers,

$$\{x_1, x_2, x_3, \ldots, x_{2025}\}$$

    such that

$$x_1 + 2x_2 + 3x_3 + \ldots 2024x_{2024} + 2025x_{2025} = 2025.$$

    Compute the minimum possible value of

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 \cdots + 2024x_{2024}^2 + 2025x_{2025}^2.$$

    Answer: $\boxed{\frac{2025}{1013}}$ *(Proposed by: Neil Dixit)*

17. Let $ABCD$ be a cyclic quadrilateral where $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Let P be the intersection of diagonals AC and BD. Calculate the area of $\triangle APD$.

    Answer: $\boxed{\frac{24\sqrt{6}}{35}}$ *(Proposed by: Neil Dixit)*

18. Consider the set of all nondegenerate triangles with integer angles. Compute the number of obtuse triangles divided by the number of acute triangles.

    Answer: $\boxed{\frac{44}{15}}$ *(Proposed by: Neil Dixit)*

Set 7 [17]

19. Consider $\triangle ABC$ with $AB = 13, BC = 14$, and $CA = 15$. Let M be a point on BC such that $\angle BAM = \frac{1}{4}\angle BAC$. Compute BM.

    Answer: $\boxed{\frac{338-39\sqrt{65}}{7}}$ *(Proposed by: Neil Dixit)*

20. Randomly choose 3 numbers (with replacement) from the set $\{1, 2, 3, \ldots, 44, 45\}$. What is the probability that these numbers can be the sides of a nondegenerate triangle?

    Answer: $\boxed{\frac{1013}{2025}}$ *(Proposed by: Neil Dixit)*

21. Josh is terrible at playing poker, so let's see if he can redeem himself. A casino offers him a deal. If he pays x grams of gold upfront, he can repeatedly roll a die, and whatever he rolls, he adds that to his total score, which starts at 0. After the first roll, he get as much gold as his score. Then on the next roll, he gets $\frac{1}{\phi}$ of his total score. The roll after, he gets $\frac{1}{\phi^2}$ of his total score and so on for an infinite number of rolls, because he is addicted to gambling. What is the minimum integer value of x such that the casino expects to make money? Note: ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2}$.

    Answer: $\boxed{24}$ *(Proposed by: Neil Dixit)*

Set 8 [18]

22. What is the remainder when

$$\sum_{w=1}^{2026}\sum_{x=1}^{2026}\sum_{y=1}^{2026}\sum_{z=1}^{2026}\left\lfloor\frac{wxyz}{2027}\right\rfloor$$

    is divided by 1000?

    Answer: $\boxed{675}$ *(Proposed by: Neil Dixit)*

23. Angela is at position $x = 0$, and moves randomly. Every minute, she either moves forward by 1 with probability $\frac{1}{2}$, backward by 1 with probability $\frac{1}{3}$, or stays put with probability $\frac{1}{6}$. At $x = 0$, she can't move backwards, so she stays still with probability $\frac{1}{2}$ instead. What is the expected number of moves it takes Angela to reach the position $x = 3$?

    Answer: $\boxed{\frac{86}{9}}$ *(Proposed by: Neil Dixit)*

24. Compute

$$\prod_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right)+\cos\left(\frac{2\pi k}{7}\right)+\tan\left(\frac{2\pi k}{7}\right)\right).$$

    Answer: $\boxed{\frac{281}{8}}$ *(Proposed by: Neil Dixit)*

Set 9 [19]

25. The Γ (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, and it satisfies $\Gamma(n) = (n-1)!$ for positive integers. For example, $\Gamma(8) = 7! = 5040$. Define $f(x)$ for positive integer x as follows:

$$f(x) = \sum_{n=1}^{x} \left(1 - \Gamma\left(2 - \frac{1}{n}\right)\right) - e.$$

Estimate the maximum x such that $f(x) < 0$.

Answer: $\boxed{1729}$ *(Proposed by: Neil Dixit)*

26. As stated in Problem 25, the Γ function extends the definition of the factorial to all positive real numbers (among others). Estimate

$$\frac{\Gamma(\pi+1)}{\Gamma(e-1)-\phi} - e.$$

Answer: $\boxed{0.001606}$ *(Proposed by: Neil Dixit)*

27. 2 prime numbers are considered "cousin primes" if the difference between them is 4. Estimate the number of cousin prime pairs under 2025^2.

Answer: $\boxed{27219}$ *(Proposed by: Neil Dixit)*

Set 10 [20]

28. Let S denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in S.

Answer: $\boxed{932.50}$ *(Proposed by: Neil Dixit)*

29. Estimate the number of primes under 10^9 that are 1 more than a multiple of 109.

Answer: $\boxed{471020}$ *(Proposed by: Alex Tsagaan)*

30. The Fermat Numbers are defined as $F_n = 2^{2^n} + 1$. Estimate the total of appearances of the digit 1 across $F_1, F_2, F_3, \ldots, F_{24}, F_{25}$.

Answer: $\boxed{202179}$ *(Proposed by: Alex Tsagaan)*