Division B

Set 1

1. Farmer John has been in the cow business for decades, but he has now gotten bored of them. He now has a farm with some numbers of unicorns and peacocks. If the total number of mouths that the farmer needs to feed is 24 and the total number of legs is 74, how many peacocks does the farmer have? Note: unicorns have 4 legs and peacocks have 2 legs.
2. Using a combination of pennies, nickels, dimes, and quarters, what is the largest total amount of money in dollars you can have without being able to create exactly one dollar using any combination of those coins?
3. Compute the remainder when 11^{73} is divided by 100.

Set 2

4. Given a right triangle with sides a, b, and c, and circumradius R, compute

$$\frac{a^2+b^2+c^2}{R^2}.$$

Note: The circumradius is defined as the radius of the circle that goes through all 3 points of a triangle.

5. Compute the sum of all positive integers that are equal to 700 times the sum of their digits.
6. How many points with integer coordinates (x, y) in the first quadrant satisfy $xy + 3x + 7y = 79$?

Set 3

7. Let α, β, γ, and δ be the roots of the polynomial $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$. Compute

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}+\frac{1}{\delta+1}.$$

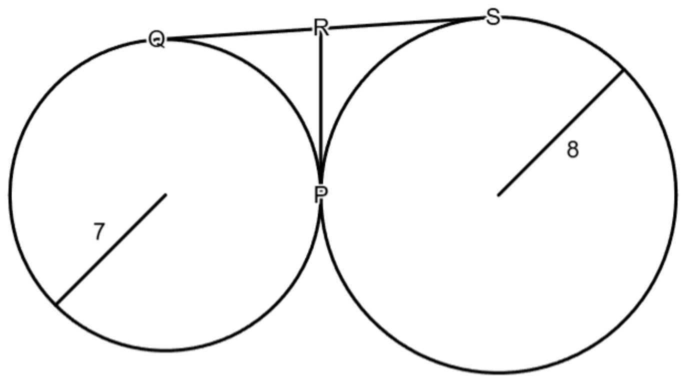
Note: The roots of an equation or polynomial is another term for its solutions.

8. A rhombus of sidelength 7 has two diagonals, and one of them is 7 times longer than the other. What is the area of the rhombus?
9. Neil wants to arrange the integers from 1 to 2025 such that if the set of all multiples of 3 and 5, are in order. For example, the number 50 must come after 45 which must come after 42, but the number 46 can be placed anywhere. In how many ways can Neil accomplish this?

Set 4

10. Let $f(x) = x^4 - px^3 + 45x^2 - qx + r$ have 4 real, positive roots where p,q,r and s are real numbers. Consider the minimum possible value of $f(1)$. It can be expressed in the form $\frac{a}{b} - c\sqrt{d}$, where a, b, c and d are positive integers such that a and b are relatively prime, and d is square-free. Compute $a + b + c + d$.

11. Suppose we have two circles of radii 7 and 8 that are externally tangent at point P. There is another common tangent line to the circles at point Q and S respectively. Let R denote the intersection of the tangent through P with QS. Compute $\overline{PR}$.

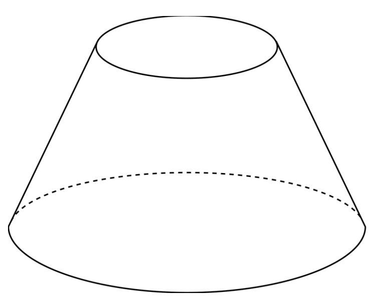


12. Calculate the number of positive integers less than or equal to 7! that are not divisible by any of $2, 3, 4, 5, 6$, or 7.

Set 5

13. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights a and b with $b > a$ below the apex. Given that the volume of the frustum equals its height, compute the minimum possible value of b.

    Note: A frustum is shaped like a sandbucket. It consists of a large cone, but an upper conical part that is similar to the whole cone has been removed. A generic example is below:



14. The ζ (zeta) function is defined as $\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}$. Compute the following sum:

$$\sum_{k=1}^{\infty} \left(\zeta\left(2k\right) - 1\right).$$

15. Alex chooses each of 2025 integers, $\{x_1, x_2, ..., x_{2025}\}$ as a random integer from 1 to 2025 (with replacement). Compute the expected number of distinct x_i that Alex gets.

Set 6

16. Suppose we have a list of positive numbers,
$$\{x_1, x_2, x_3, ..., x_{2025}\},$$
such that
$$x_1 + 2x_2 + 3x_3 ... + 2024x_{2024} + 2025x_{2025} = 2025.$$
Compute the minimum possible value of
$$x_1^2 + 2x_2^2 + 3x_3^2 ... + 2024x_{2024}^2 + 2025x_{2025}^2.$$

17. Let $ABCD$ be a cyclic quadrilateral where $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Let P be the intersection of diagonals AC and BD. Calculate the area of $\triangle APD$.

18. Consider the set of all such nondegenerate triangles. Compute the number of obtuse triangles divided by the number of acute triangles.

    Note: a nondegenerate triangle is a fancy way of saying that the triangle has nonzero area.

Set 7

19. Consider $\triangle ABC$ with $\overline{AB} = 13, \overline{BC} = 14, \overline{CA} = 15$. Let M be a point on BC such that $\angle BAM = \frac{1}{2}\angle BAC$. Compute $\overline{BM}^2$.

20. Suppose that you choose three integers (with replacement) from the set $\{1, 2, 3, ..., 44, 45\}$. How many nondegenerate triangles can you create?

21. Josh is terrible at playing poker, so let's see if he can redeem himself. A casino offers him the following deal:

    If he pays x grams of gold upfront, he can repeatedly roll a die, and whatever he rolls, he adds that to his total score, which starts at 0. After the first roll, he get as much gold as his score. Then on the next roll, he gets $\frac{1}{\phi}$ of his total score. The roll after, he gets $\frac{1}{\phi^2}$ of his total score and so on for an infinite number of rolls, because he is addicted to gambling. What is the minimum integer value of x such that the casino expects to bankrupt Josh in the long run? Note: ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Set 8

22. What is the remainder when the following sum,
$$\sum_{w=1}^{2026}\sum_{x=1}^{2026}\sum_{y=1}^{2026}\sum_{z=1}^{2026}\left\lfloor \frac{wxyz}{2027} \right\rfloor,$$
is divided by 1000.

    Note: The floor function ($\lfloor\ \rfloor$) denotes the smallest integer less than or equal to the value inside.

23. Angela is at position $x = 0$, and he moves randomly. An event consists of either moving forward by 1, backward by 1, or staying still. For any event, she moves forward with probability $\frac{1}{2}$, backwards with probability $\frac{1}{3}$, and stays still with probability $\frac{1}{6}$. What is the expected number of moves it takes Angela to reach the position $x = 3$? At $x = 0$, she can't move backwards, and stays still with probability $\frac{1}{2}$ instead.

24. Compute
$$\prod_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right) + \cos\left(\frac{2\pi k}{7}\right) + \tan\left(\frac{2\pi k}{7}\right)\right).$$
(This problem cannot be solved, and only exists to prevent a team sweeping. Don't spend time on it!)

Set 9

25. The Γ (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, where $\Gamma(n) = (n-1)!$. E.g. $\Gamma(8) = 7! = 5040$. Define $f(x)$ for positive integers, x, as follows:

$$f(x) = \sum_{n=1}^{x} \left(1 - \Gamma\left(2 - \frac{1}{n}\right)\right) - e.$$

Estimate the maximum x such that $f(x) < 0$.

26. As stated in Problem 25, the Γ function extends the definition of a factorial to all positive real numbers. Estimate

$$1000\left(\frac{\Gamma(\pi+1)}{\Gamma(e+1)} - \phi\right),$$

where ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2}$.

27. 2 prime numbers are considered "cousin primes" if the difference between them is 4. Estimate the number of cousin prime pairs under 2025^2.

Set 10

28. Let S denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in S.

29. Estimate the number of primes p under 1 billion such that p is 1 more than a multiple of 109.

30. The Fermat Numbers are defined as

$$F_n = 2^{2^n} + 1.$$

Determine the total of appearances of the digit 1 across $F_1, F_2, F_3, ..., F_{25}$.