Division A

Set 1

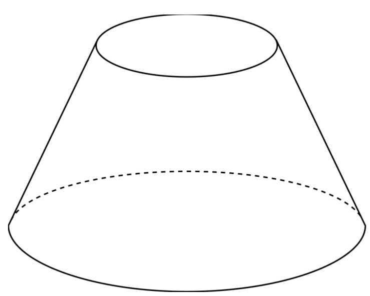
1. How many points with integer coordinates, (x, y), in the first quadrant satisfy $xy + 3x + 7y = 79$?
2. A rhombus of side length 7 has two diagonals, one which is 7 times longer than the other. What is the area of the rhombus?
3. Find the remainder when 11^{73} is divided by 100.

Set 2

4. Suppose we have two circles of radii 7 and 8 that are externally tangent at point P. There is another common tangent line to the circles at point Q and S respectively. Let R denote the intersection of the tangent through P with QS. Compute $\overline{PR}$.
5. Given that
$$\sqrt{x+2} + \sqrt{x+5} = 5,$$
compute x.
6. Ayush is in the mood for some fried chicken. KFC serves chicken tenders in three sizes: boxes of 4 pieces, 7 pieces, and 11 pieces. Find the maximum number of chicken tenders that Ayush cannot buy at KFC.

Set 3

7. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights a and b with $b > a$ below the apex. Calculate the minimum b such that there exists some a such that the frustum's volume equals its height. Note: A frustum is what is left of a larger cone if you remove a smaller cone similar to the larger cone from the top, and looks like a sandbucket. A generic frustum is shown below:



8. Mary chooses three integers, with replacement, from the set $\{1, 2, 3, ..., 44, 45\}$. How many distinct nondegenerate triangles can she create?
9. Let α, β, γ, and δ be the roots of the polynomial $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$. Compute
$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} + \frac{1}{\delta+1}.$$

Set 4

10. Alex has a set of positive numbers, $\{x_1, x_2, x_3 ... x_{2025}\}$, such that

$$x_1 + 2x_2 + 3x_3 ... + 2024x_{2024} + 2025x_{2025} = 2025.$$

Calculate the minimum possible value of

$$x_1^2 + 2x_2^2 + 3x_3^2 ... + 2024x_{2024}^2 + 2025x_{2025}^2.$$

11. Consider $\triangle ABC$ with $\overline{AB} = 13, \overline{BC} = 14, \overline{CA} = 15$. The angle bisector of $\angle BAC$ meets the side at M. Compute $\overline{AM}^2$.

12. Samuel is a nerd who likes coding and discrete math. He is thinking about the set of all triangles with integer angles. He is trying to run code to figure out the number of obtuse triangles divided by the number of acute triangles. Show him how to be the better nerd and compute it for him.

Set 5

13. Calculate the following sum:

$$\sum_{n=0}^{\infty} \frac{\cos(n)}{2^n}.$$

14. Let $ABCD$ be a cyclic quadrilateral where $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Let P be the intersection of diagonals AC and BD. Calculate the area of $\triangle APD$.

15. Alex randomly chooses eight distinct integers $a_1 < a_2 < ... < a_8$ from 1 to 2024. Find the expected value of a_3.

Set 6

16. Let N be the answer to Problem 18. Let ω_1 be a circle centered at O, and let A be a point in the interior of ω_1. Let B be a point on ω_1 such that $\overline{AB} \perp \overline{OA}$. Consider the locus of all points S such that the circle with center S passing through A is tangent to ω_1. Given that the ratio of the enclosed area of S to the area of ω_1 is N, find the ratio of the area of ω_1 to the area of the circle with diameter AB.

17. Let N be the answer to Problem 16. For some real $0 < k < 1$, consider the graphs of $N\cos(x)\sin(y) = k$, and $\sin(x)\sin(y) = k$ for $0 < x, y < \pi$. Find the largest k such that these two graphs intersect.

18. Let $\frac{a}{\sqrt{b}}$ be the answer to Problem 17. John has a drawer of $\frac{b+2}{3}$ socks, each with a specific color, and exactly a of the socks are red. Given that the number of socks of each color are distinct, find the maximum number of different colors of socks John could have.

Set 7

19. Compute the remainder when

$$\sum_{w=1}^{2026}\sum_{x=1}^{2026}\sum_{y=1}^{2026}\sum_{z=1}^{2026}\left\lfloor\frac{wxyz}{2027}\right\rfloor$$

is divided by 1000.

20. Jerry really likes dodecahedrons and spheres. Let r denote the radius of the largest sphere that can be inscribed in a unit dodecahedron. r^2 can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are relatively prime positive integers such that c is square-free and b and d are relatively prime. Compute $a + b + c + d$.

21. Let a, b, and c be the roots of $x^3 - kx^2 + 69x + 15$. Given that there exist nonzero complex numbers, x, y, and z such that

$$\begin{cases} x = (y+z)(a+1) \\ y = (z+x)(b+1) \\ z = (x+y)(c+1), \end{cases}$$

find k.

Set 8

22. Let acute triangle ABC with $\angle B < \angle A < \angle C$ have orthocenter H and circumcenter O. Given that HO is parallel to AC, and $\tan A = 4$, find $\tan B$.

23. Angela is at position $x = 0$, and she moves randomly. An event consists of either moving forward by 1, backward by 1, or staying still. For any event, she moves forward with probability $\frac{1}{2}$, backwards with probability $\frac{1}{3}$, and stays still with probability $\frac{1}{6}$. At $x = 0$, Angela can't move backwards, and stays still with probability $\frac{1}{2}$ instead. What is the expected number of events it takes her to reach the position $x = 4$?

24. Compute
$$\prod_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right)+\cos\left(\frac{2\pi k}{7}\right)+\tan\left(\frac{2\pi k}{7}\right)\right).$$

Set 9

25. The Γ (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, where $\Gamma(n) = (n-1)!$. E.g. $\Gamma(8) = 7! = 5040$. Define $f(x)$ for positive integers, x, as follows:
$$f(x) = \sum_{n=1}^{x}\left(1-\Gamma\left(2-\frac{1}{n}\right)\right)-e.$$
Estimate the maximum x such that $f(x) < 0$.

26. As stated in Problem 25, the Γ function extends the definition of a factorial to all positive real numbers. Estimate
$$1000\left(\frac{\Gamma(\pi+1)}{\Gamma(e+1)}-\phi\right),$$
where ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2}$.

27. 2 prime numbers are considered "cousin primes" if the difference between them is 4. Estimate the number of cousin-prime pairs under 2025^2.

Set 10

28. Let S denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in S.

29. Estimate the number of primes p under 1 billion such that p is 1 more than a multiple of 109.

30. The Fermat Numbers are defined as
$$F_n = 2^{2^n} + 1.$$
Determine the total of appearances of the digit 1 across $F_1, F_2, F_3, ..., F_{25}$.