GUNN MATH COMPETITION

Team Round A

Division A // 60 Minutes // March 24TH, 2024

1. (20 points) If $f(x, y) = 3x^2 + 3xy + 1$ and $f(a, b) + 1 = f(b, a) = 42$, then determine $|a + b|$.

2. (20 points) Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR?

3. (25 points) How many (possibly empty) subsets of $\{1, 2, \cdots, 10\}$ have a sum that is at most 27?

4. (25 points) Let S_1 be the set of all integers that can be expressed in 2^a3^b, where $a + b$ is even, and let S_2 be all integers that can be expressed in 2^a3^b, where $a + b$ is odd. What is

$$\left(\sum_{s\in S_1} \frac{1}{s}\right) - \left(\sum_{s\in S_2} \frac{1}{s}\right)?$$

5. (30 points) There are 10 chairs ordered in a line at the doctor's office. Due to social distancing rules, any pair of people must be separated by at least 1 chair. How many ways can any number of people sit in the chairs while maintaining social distancing?

6. (30 points) How many base-10 numbers have all nonzero digits that sum to 12?

7. (35 points) In triangle ABC, D, E, and F are points on BC, AC, and AB such that AD, BE, and CF intersect at X. Additionally, $AX = DX$ and $BX = 4EX$. Let $[ABC]$ denote the area of a triangle ABC. Compute $\frac{[DEF]}{[ABC]}$.

8. (40 points) Four factors a, b, c, and d of 648 (not necessarily distinct) are randomly chosen. What is the probability that $\gcd(a, b) = \operatorname{lcm}(c, d)$?

9. (45 points) Suppose that Josh has 27 cubes. At random, he selects an integer a between 0 and 27, inclusive, and he colors a of the cubes red and the remaining cubes blue. He then randomly arranges the cubes into a larger $3 \times 3 \times 3$ cube. Suppose that the central cube is red. What is the expected number of corner cubes that are also colored red?

10. (50 points) Samuel begins at the origin and picks a random angle $0° < \theta < 360°$. Then, on the n-th second, he moves forward by $\left(\frac{1}{\sqrt{2}}\right)^{n-1}$ meters and turns θ degrees counterclockwise. He eventually converges on a point in the plane. What is the probability that this point is no more than $\sqrt{2}$ units away from the origin?

11. (50 points) Let $ABCD$ be a square of side length 6. Suppose O is the center of the square, and E is the midpoint of CD. Let P be the set of points X such that $OX = 2EX$. The outline of P divides $ABCD$ into two regions. What is the area of the smaller region?

12. (55 points) A regular 10-gon $P_1P_2 \cdots P_{10}$ is inscribed in a circle of radius 1. Let $a_{i,j}$ be the distance between points P_i and P_j. What is

$$\prod_{1\le i<j\le 10} a_{i,j}?$$

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Team Round A (continued)

13. (60 points) Let a, b, and c be complex numbers such that

$$\begin{aligned} a+b+c &= 4, \\ a^2+b^2+c^2 &= 12, \\ a^3+b^3+c^3 &= 64. \end{aligned}$$

Let n be the largest integer for which 2^n divides $a^{2024}+b^{2024}+c^{2024}$. Compute n.

14. (65 points) Roger is standing on the point $(0, 0)$ on the coordinate plane and wants to get to $(4, 4)$. He can only move one unit at a time and starts by facing to the right. On each of his moves, he can either take a step forward or turn 90 degrees counterclockwise and then take one step forward. If Roger never steps out of the square bounded by $|x| \leq 4$ and $|y| \leq 4$ and never visits the same point twice, how many paths can he take?

15. (80 points) Steve hops around the integers on the number line. He starts at 0, and every second, he moves back 1 unit with probability $\frac{1}{2}$, and he moves forward $n > 0$ units with probability $\frac{1}{2^{n+1}}$. Eventually he lands on an integer he has already visited. On average, how many seconds does it take for this to occur?