GUNN MATH COMPETITION

Individual Round A

Division A // 60 Minutes // March 24TH, 2024

1. Let $ABCDEF$ be a hexagon with unit side length. Find the ratio of the area of the intersection of $\triangle ACE$ and $\triangle BDF$ to the area of the hexagon $ABCDEF$.

2. Suppose a, b, and c are real numbers such that

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)=\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$$

   If $abc = 11$, what is $a+b+c$?

3. For each of the next 5 days, the chance of rain is $\frac{1}{2^n}$, where n is the number of days that it has already rained. (It will always rain on the first day.) What is the probability that in the next 5 days, it rains for 4 days?

4. Suppose a and b are two integers such that $\frac{ab+1}{a-2b} = 11$. What is the maximum possible value of ab?

5. In right triangle ABC, let M be the midpoint of hypotenuse BC. Suppose the incircle of triangle ABM is tangent to AM at X, while the incircle of triangle ACM is tangent to AM at Y. If $BC^2 = 218$ and $XY = 3$, compute the area of triangle ABC.

6. Let $P(x)$ be a degree 98 polynomial such that $P(a) = \frac{a^2+1}{a}$ for all integers a between 1 and 99, inclusive. Compute $P(100)$.

7. Find the number of unordered sets $\{a, b, c\}$ of three positive integers for which

$$abc = 27000.$$

8. Compute the remainder when the following expression is divided by 103:

$$\prod_{n=1}^{103}(n^4 - n^3 + n^2 - n + 1).$$

9. A square of side length 1 is rotated clockwise about its center, by some angle θ less than 45 degrees. Suppose that the intersection of the original square and the rotated square has area $\frac{5}{6}$. Compute $\tan\theta$.

10. Angela writes the integers 1 through 31 in increasing order on a whiteboard. Ethan then randomly selects a real number x between 0 and 32 (assume that x is not an integer). He wants to insert x into the correct position in Angela's list, maintaining the order of the list.

    To achieve this, Ethan follows the process of binary search. First, he compares x to the central number in Angela's list. If x is smaller, he narrows his search to the beginning 15 integers, and otherwise he narrows his search down to the final 15 integers. He continues to narrow down the list until just one integer remains. By comparing x to this final integer, he decides whether to insert x directly before or after the integer.

    However, right before Ethan begins his search, Angela randomly swaps 1 with another integer written on the whiteboard. After Ethan is finished, Angela undoes the swap. What is the probability that the list is now sorted?