Division A Set 1 $\left(4\ \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Let point M be on segment BC of $\triangle ABC$ so that $AM = 3$, $BM = 4$, and $CM = 5$. Find the largest possible area of $\triangle ABC$.
2. 7 distinct lines intersect at n distinct points. Find the product of all possible values of n.
3. How many times per day do the minute and hour hand of a clock coincide?

Division A Set 2 $\left(5\ \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Square $ABCD$ has side length 10. Point E is on BC such that $BE = 6$. Point F is on AD such that the ratio of areas $ABEF$ and $ECDF$ is 2. Find FD.
2. How many ways are there to color a 2×2 grid with 4 colors, such that no two cells that share an edge have the same color? Rotations and reflections are considered distinct.
3. For how many integers n is $\frac{n}{30-n}$ the square of an integer?

Division A Set 3 $\left(6\ \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Find the number of ways to arrange the numbers $1, 2, \ldots, 8$ such that no two adjacent numbers share a prime factor.
2. A divisor d of a number is *unitary* if it has the property $\gcd\left(d, \frac{n}{d}\right) = 1$. What is the sum of all unitary divisors of 1620?
3. Evaluate the infinite fraction

$$F = \frac{1}{(1-a_1)\frac{1}{(1-a_2)\frac{1}{1-\cdots}}},$$

where $a_i = i$, if $i > 1$ and divides 14 and $a_i = 0$ otherwise.

Division A Set 4 $\left(7\ \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Call a pair of positive integers (a, b) with $a > 2$ *nice* if for all numbers $\overline{wxyz}_a + \overline{wx}_a \equiv \overline{yz} \pmod{b}$. Find the sum of b across all *nice* pairs (a, b) such that $a < 10$.
2. What is $\sum_{n=0}^{\infty} \frac{5^n + 5^{n-1}4^1 + 5^{n-2}4^2 + \cdots + 4^n}{20^n}$
3. Let C be the sphere $x^2 + y^2 + (z-1)^2 = 1$. Point P on C is $(0, 0, 2)$. Let $Q = (14, 5, 0)$. If PQ intersects C again at Q', then find the length PQ'.

Division A Set 5 $\left(8 \, \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Rectangle $ABCD$ has $AB = 2$ and $BC = 4$. Initially, the rectangle lies flat on the ground. Then, vertex C is held 2 units off the ground while vertex A is fixed in place so that $ABCD$ can rock back and forth with AC as the axis of rotation. The total angle that $ABCD$ can rotate is θ. Compute $\tan\theta$.

2. Brandon the painter wants to paint five consecutive houses on a street. He has red, blue, and yellow paint, but he is not allowed to paint two adjacent houses yellow and blue. In how many ways can he paint the five houses?

3. In a non-square rectangle, construct the diagonals, and for each pair of midpoints of the sides, draw a line between them. This should divide the rectangle into 16 smaller triangles. Using the constructed line segments, how many resulting similar triangle pairs are there?

Division A Set 6 $\left(10 \, \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Let $f(x) = \lfloor \frac{x}{2.7} \rfloor$. Find the sum of all integers a such that $f(f(f(a))) = 1$.

2. Suppose a faulty coin flips heads $\frac{1}{3}$ of the time and tails $\frac{2}{3}$ of the time. What is the probability that you land heads 3 times before landing tails 3 times?

3. Consider a unit cube in 7 dimensions. A diagonal is defined as any line between two corners that are not connected by an edge. How many diagonals are at least 2 in length?

Division A Set 7 $\left(12 \, \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Mary has 2024 stones. Every day, she picks a random integer from 1 to 2024 and takes away that number of stones (or as many as she can until the pile runs out). Let $\frac{m}{n}$ be the expected number of days she takes to run out of stones. Find the sum of the distinct prime factors that divides m.

2. Let $a(n)$ be the average value of the divisors of n. For example, $a(6) = \frac{1+2+3+6}{4} = 3$. Compute
$$\sum_{d|29400} a(d).$$

3. How many distinct colorings can be obtained when using two colors to color the faces of a cube? Two colorings are identical if they can be obtained from each other by rotation (but not reflection).

Division A Set 8 $\left(15 \, \frac{\textbf{points}}{\textbf{problem}}\right)$

1. Sophie begins at the origin. Each of her steps is either 1 unit right, 1 unit up, or 1 unit both right and up (i.e. $(1, 0)$, $(0, 1)$, or $(1, 1)$). Let N be the number of distinct sequences of steps that she can take to end up at $(101, 101)$. Compute the remainder when N is divided by 101.

2. Suppose that f is a function with $f(1) = 1$, $f(2n) = f(n)$, and $f(2n + 1) = f(n + 1) + f(n)$. Compute
$$\sum_{a=1}^{255} f(a).$$

3. For a triangle ABC, $AB = 1$, $BC = 1 + \sqrt{5}$, and B equals 60 degrees. If D is the midpoint of line AC, what is $\tan(ABD)$?