Division A Team Problems

1. (20 points) What is the sum of all x such that $x^{x-2} = x^3$?

2. (20 points) A rectangular prism is formed with side lengths x, y, z. A triangle is formed using side lengths $\log x$, $\log y$, and $\log z$ (where log is in base 10). If the volume of the prism is 100, find the perimeter of the triangle.

3. (25 points) What is the area of the triangle with side lengths equal to the roots of the equation $x^3 - 12x^2 + 47x - 60$?

4. (25 points) The increasing sequence $1, 5, 6, 25, 26, 30, 31, ...$ consists of all positive integers that are powers of 5 or sums of distinct powers of 5. What is the 50th number of this sequence?

5. (30 points) Let $p > 0$ be the probability that a coin lands heads. The probability that the coin lands heads 1776 of 2023 times is the same as if the coin lands 1777 times. Find p.

6. (30 points) A sample of 4 M&Ms is taken from an infinite supply of M&Ms that is $\frac{1}{3}$ blue and $\frac{2}{3}$ red. Two M&Ms are randomly drawn with replacement from this sample, and both times a red M&M is drawn. What is the expected number of red M&Ms in the sample?

7. (35 points) Let $\{b\}_n$ be the sequence of terms defined by $b_{2n-1} = 3n - 2$ and $b_{2n} = 3n - 1$. Additionally, let $a_n = b_1 + b_2 + \cdots + b_n$. How many terms between a_1 and a_{42}, inclusive, are divisible by 5?

8. (40 points) Let $a_0 = 1$ and $\frac{a_{n+1}}{n+1} = \left(1 + \frac{1}{n}\right)^n a_n$ for all $n \geq 0$. Find the least p such that $a_p \equiv p \pmod{23}$.

9. (40 points) Let $f(x)$ denote the number of trailing zeros in $x!$. For how many 3 digit integers x is $f(x+5) - f(x)$ an odd number?

10. (45 points) Let $S = \{1, 2, ..., 10\}$. S is randomly split into 5 subsets, all of size two. Let $S_1 = \{1, 2, ..., 5\}$ and $S_2 = \{6, 7, ..., 10\}$. Let k be the expected number of two-element subsets with one element in S_1 and one element in S_2. Compute k.

11. (50 points) How many perfect squares from 1^2 to 1000^2 are within 2 of a multiple of 7 and within 1 of a multiple of 11?

12. (55 points) Let P be a point in the coordinate system. Every second, we reflect the point P over another point. At the nth second with n starting at 0, reflect P over $\left(\frac{1}{2^n}, \frac{1}{3^n}\right)$. Eventually, the point approaches $(1, 1)$ and $(-1, -1)$ on every other turn, respectively. Find the maximum possible sum of the original coordinates of P.

13. (55 points) Define $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ and $g(x) = \left\lceil \frac{x}{2} \right\rceil$. Let s be a length 7 string consisting of the functions f and g. When the functions of s are applied to 197 in order from start to finish, the value 1 is outputted. How many possible strings s are there?

14. (60 points) In still water, Daisy the dolphin can swim at a maximum speed of 3m/s. Daisy wants to swim across a river of width 20m to the location directly across from where she starts. For the first 5 seconds she swims, there is a current of 1m/s along the river. After 5 seconds, a sudden storm reverses the current and the river begins flowing in the opposite direction at 2m/s. what is the minimum time, in seconds, it takes for her to swim across the river?

15. (70 points) 71 students sit in a row of 71 chairs. A deck of 71 cards numbered from 1 to 71 is shuffled, and a card is taped to each of the 71 chairs. Every second, each student looks at the number of the card on their chair, and moves to the chair with that number. At the beginning, 23 students are chosen. After 71 seconds, let the number of chairs that have been sat on by a chosen student be C. What is the expected value of C?