Division A Individual Problems

1. What is the largest prime factor of $3^{12} - 2^{12}$?

2. Let x be a 3-digit number in base b, and let y be a 2-digit number in the same base b. Given that $x - y = 1032_{10}$, and $x + y = 1150_{10}$, find the sum of all possible values of b.

3. Electrifying Ezra the Eastern Electric Eel has 4 electric light switches, all initially switched *off*. Every second, he randomly chooses an integer k from 1 to 4, inclusive. Then, he randomly chooses k of the 4 switches and toggles each one. After 4 seconds, find the expected number of switches that are *on*.

4. A fly is tethered to a vertex of a solid cube of side length 2023 by a freely moving rope of length 6. Let S be the region in space that the fly can reach. Find the absolute difference between the volume and surface area of S.

5. Let $y > x > 0$. Given that $\log y = 2 \log x + \log \log x$ and $x^3 (\log x)^2 = 4y + 5x$, find $\sqrt[x]{10^y}$. (Here $\log x$ is base 10.)

6. A bus driver is driving along a one-dimensional number line. He starts at the origin, and his end destination is at the point 7. Along the way, he must stop at the points 1 through 6 in some randomly chosen order. What is the expected distance that the bus travels in total?

7. There are 100 lights numbered 1 to 100, all initially off. For factor k of 100, the state of every light numbered a multiple of k alternates. For example, when $k = 50$, the state of lights 50 and 100 would be toggled. After doing this for all factors of 100, how many lights are on?

8. Let D be a point on side BC of triangle ABC such that triangle BAD is similar to triangle BCA. Suppose that the angle bisector of $\angle B$ intersects AD at E and AC at F. Given that $AB = 5$, $BE = 4$, and $\angle ABC = 60^\circ$, find AF^2.

9. Find the largest power of 2 that divides

$$\sum_{n=0}^{2023} n(n+1)(n+2)(n+3)(n+4).$$

10. A parallelogram with side lengths 8 and 10 has angles 60° and 120°. A rhombus is inscribed so that each side of the parallelogram (not including endpoints) contains one vertex of the rhombus. Given that the vertices of the rhombus divide the sides into integer-length parts, find the sum of the squares of all possible side lengths of the rhombus.