Division A SET 1 - 4 points per problem

1. What is $i + 2i^2 + 3i^3 + 4i^4 + ... + 64i^{64}$?

2. A bag contains 2 red balls and 3 blue balls. What is the probability that two consecutive draws (without replacement) from this bag will result in two balls of the same color? Express your answer as a common fraction.

3. The average of five consecutive even integers is 90. What is the average of the smallest three numbers of those integers?

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Division A SET 2 - 5 points per problem

4. Four integers a,b,c, and d are randomly chosen with replacement from between 0 and 2023, inclusive. What is the probability that $(a - d)(b - c)$ is even?

5. Ayush the astronaut is still on a mission to find "Who asked?". Unfortunately, Ayush has just discovered that 4 small asteroids have punctured the spaceship. Fortunately, Ayush is in possession of an Advanced Self-Sustaining Bio-Regenerative Oxygenated Atmosphere Production Unit™, a device that can produce an arbitrary amount of oxygen. Given that the 4 holes are emptying air at a rate of 5 minutes per liter, 6 minutes per liter, 10 minutes per liter, and 30 minutes per liter, respectively: What is the minimum speed, in minutes per liter, that Ayush should set the Atmosphere Production Unit™ such that he does not run out of air?

6. 119 consecutive odd numbers sum to 2023. What is the difference between the largest and smallest number?

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Division A SET 3 - 6 points per problem

7. Allan has 100 coins, some of which have a weighted with a probability 0.6 landing heads while the rest are fair. If the expected number of heads after tossing all 100 coins is 59, how many fair coins does Allan have?

8. 2023^{2023} is fully expanded and written down on a large whiteboard. The sum of digits is calculated. Then, the sum of the digits of the result is calculated, and so forth until we are left with a single digit. What is that digit?

9. How many four-digit even positive integers have four different digits, with 5 being the largest digit?

Division A SET 4 - 7 points per problem

10. Compute $3\sqrt{9\sqrt{27\ldots}}$, where each subsequent square root contains the next power of 3.

11. On a 6 by 6 grid of points, every point is to be colored as one of the three colors: yellow, green, or blue. What is the expected number of rectangles with all four vertices having the same color? (We only count rectangles having one pair of sides parallel to the x-axis and the other pair parallel to the y-axis).

12. Roger has a regular die with 6 sides. He rolls the die until he rolls a 1, a 5, or two consecutive rolls with even numbers. Compute the expected number of rolls Roger will need to achieve his goal.

Division A SET 5 - 8 points per problem

13. An equilateral triangle of side length $\frac{2}{3}$ is dropped onto an infinite grid of unit squares. Given that the triangle lands such that one of its sides is parallel to one of the axes of the grid, what is the probability that the triangle doesn't intersect any grid lines?

14. Find A such that $0 < A < 90^\circ$ and $\cos(34^\circ) + \sin(34^\circ) = \sqrt{2}\sin(A)$.

15. Jinwoo and Steve are playing a coin game. The game works as follows:

    - The players take turns flipping coins.
    - The winner of the game is the player that flips the first head.
    - The winner of the previous game flips first.

    Assuming Jinwoo flips first in the first game, What is the probability that Jinwoo wins the 5th game?

Division A SET 6 - 10 points per problem

16. Let $S(n)$ denote the sum of the factors of an integer n. For how many composite positive integers $n < 100$ is $S(n)$ divisible by 7?

17. A calculator has two buttons, $\times 2$ and $+1$. Initially, it displays 1 on the screen. Clicking a button will apply that function to the displayed number immediately.

    This calculator also has a special rule that *twice* the number of "+1"s that have been applied may at no time exceed the number of "x2"s applied. How many numbers fewer than 512 can be reached through some sequence of these button presses (note "1" can also be achieved, through no button presses)?

18. The Martians have a very interesting system of geometry where the distance between points (a, b) and (c, d) is defined as $\sqrt{|(a-c)^3 + (b-d)^3|}$. For the Martians, what value of $c > 1$ would cause the points $(0, 0)$, $(1, -c)$, and $(c, -1)$ to form an equilateral triangle (an equilateral triangle on Mars is defined as three points whose pairwise distances are all equal)?

Division A SET 7 - 12 points per problem

19. Find

$$\sum_{k=0}^{\infty} \frac{2^{k-2^{k+1}}}{1+2^{-2^{k+1}}}.$$

20. In an arrangement of $10 \times 10 \times 10$ cubes, k of them are colored red. Every second, every non-red cube that is adjacent to at least 3 colored cubes is colored red. After a while, every cube is colored red. What is the minimum possible value of k? (Cubes are adjacent if they share a face).

21. Let $a \oplus b = \frac{a+b}{1+ab}$, and let

$$a \otimes n = \underbrace{a \oplus a \oplus \cdots \oplus a}_{n\ a\text{'s}}.$$

    There are 2022 complex *nonzero* solutions to $x \otimes 2023 = 0$, and let them be $x_1, x_2, \cdots, x_{2022}$. Compute

$$\sum_{n=1}^{2022} \frac{2}{1-x_n}.$$

Division A SET 8: ESTIMATION - 15 points *max* per problem

22. The chance that Alan touches grass today if he did not touch grass the previous day is 50%. However, if he did touch grass yesterday, then the chance he touches grass drops to 25%. Given that Alan touched grass today, what is the expected number of times he will touch grass over the following 23 days?

23. Every edge of a regular tetrahedron is colored one of 10 colors. If two colorings are the same if one can be rotated into the other, then how many colorings are there?

24. How many ways are there to place 23 queens on a 23×23 chessboard such that no two queens attach each other? Note: a queen attacks any number of squares vertically, horizontally, or diagonally.