GMC 2022 Team B Solutions

Grand Mega Cool ppl

March 5, 2022

1. Let $i = \sqrt{-1}$. Simplify $i^2 + i^8$.

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{0}$

   Since $i = \sqrt{-1}$, i^2 will simply be -1. Then $i^8 = (i^2)^4 = (-1)^4 = 1$ so we have that $i^2 + i^8 = -1 + 1 = 0$.

2. Evaluate 1111111^2.

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{1234567654321}$

   We can rewrite this as $(1111111)(1000000 + 100000 + 1000 + 100 + 10 + 1)$ which distributes to $(1111111000000+111111100000+11111110000+1111111000+111111100+11111110 + 1111111)$. This is then equal to 1234567654321.

3. If $a@b$ is $(a - b)(a + b)$ for positive integers a, b, what is $5@2$?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{21}$

   Using the definition for the @ function we get that $5@2 = (5-2)(5+2)$ which is equal to $3 \cdot 7 = 21$.

4. In a triangle ABC with integer side lengths and positive area, $AB = 15$ and $BC = 7$. Find the number of possible values for the length of side AC.

   Proposed by: Arul Mathur

   Answer: $\boxed{13}$

   By the triangle inequality we know that side length AC must be less than $15 + 7 = 22$ and greater than $15 - 7 = 8$ as any number 22 or above or 8 or below would result in the triangle inequality failing. Then we have all the numbers between 9 and 21 inclusive which would be $21 - 9 + 1 = 13$ numbers.

5. The units digit of a two-digit number is 3. The tens digit of that two-digit number is x and when the two-digit number is reversed, the differences of the two-digit numbers are 27. What is x?

Proposed by: Jinwoo Jeong

Answer: $\boxed{6}$

Our two digit number is $\overline{x3}$ which has a value of $10x + 3$. When reversed we get $3x$ which has a value of $30 + x$. We know the difference between the two which is $10x + 3 - (30 + x) = 9x - 27$ is equal to 27. Solving we get that $x = 6$.

6. Samuel, Jerry, Ethan, Josh, and David have all just finished a math test. Samuel, Jerry, Ethan, and Josh scored a 91, 94, 92, and 93, respectively. However, David decided that the group's average score was too high, and he did badly on the test so that the average score would be below 90. What was the highest possible score David got?

   Proposed by: Arul Mathur

   Answer: $\boxed{79}$

   Let's call David's score on the test x. Then the average score of the group is $(91 + 94 + 92 + 93 + x)/5$ which needs to be below 90. We then solve this inequality for x to get that $370 + x < 450$ so $x < 80$ and therefore the highest score that David could have gotten is 79.

7. If 15 consecutive even numbers sum to 300, what is the largest number?

   Proposed by: Andrew Peng

   Answer: $\boxed{34}$

   Since we have consecutive even numbers, we can find the middle number which is $300/15 = 20$. The largest number will be $20 + 7 \cdot 2 = 34$.

8. A magic product square is an $N \times N$ grid of squares with each square containing a positive integer such that the product of the numbers in every row, column and main diagonal of the square is the same. A certain 3×3 magic product square has this common product equal to 5832. What integer is in the middle square of the grid?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{18}$

   Consider the following magic square:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

   Let the common product be P. By the given properties, we have $aei = beh = ceg = P$, and multiplying them together, we have $aibhcge^3 = P^3$. However, we also have that $abc = ghi = P$, so $aibhcge^3 = abcghie^3 = P^2e^3 = P^3$, and so $e = \sqrt[3]{P}$. Since $P = 5832$, the cube root of 5832 is $e = 18$.

9. Find the units digit of 17^{123}.

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{3}$

   To find the units digit we will look for a pattern in the units digits of powers of 7. Note that we only care about the 7 as even though we are taking 17 to the power of 123 as we only care about the units digit, the 1 doesn't matter and we only consider the 7. Taking the first few powers of 7 we get 7, 49, 343, 2401, 16807 ... and we notice that the units digit repeats in the pattern 7, 9, 3, 1 ... Now we must find the remainder when 123 is divided by 4 to determine the units digit of 7^{123}, and doing this we get that 123 has remainder 3, so the units digit is 3.

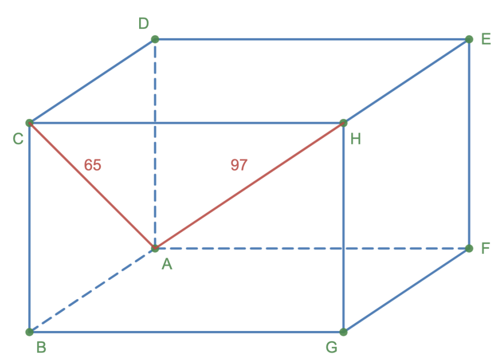
10. What is the sum of the *real* roots of the cubic polynomial $x^3 - x^2 - x - 2$?

    Proposed by: Ayush Aggarwal

    Answer: $\boxed{2}$

    We first use rational root theorem to check for any rational roots of this polynomial so we check -1, -2, 1 and 2 and we see that 2 works as a root. Then dividing out $x - 2$ we get the resulting quadratic $x^2 + x + 1$. The determinant of this quadratic is $1^2 - 4 \cdot 1 \cdot 1 = -3$ and since this is negative the quadratic has no real roots so our only root is just 2.

11. In a rectangular prism $ABCDEFGH$ shown below, the length of the space diagonal AH is 97 and the length of diagonal AC is 65. What is the length of CH?



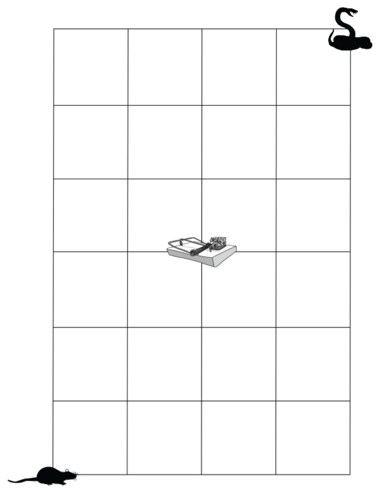
    Proposed by: Alan Lee

    Answer: $\boxed{72}$

    Looking at triangle ACH, we can see that this is actually a right triangle with right angle at C since line segments CH and CA lie on planes that are perpendicular. Then we just use the Pythagorean Theorem to get that CH $= \sqrt{97^2 - 65^2} = 72$.

12. Roger the rat wants to meet Saumya the snake for a lunchtime tea party. Roger is currently 6 blocks south and 4 blocks west of where Saumya currently is. However,

Roger despises the giant cheesetrap 3 blocks north and 2 block east of where he is currently, and will not travel through that intersection. How many paths going only north and east can Roger follow to meet Saumya?



Proposed by: Alan Lee

Answer: $\boxed{110}$

The total amount of ways that Roger can get to Saumya disregarding the mousetrap is $\binom{10}{4} = 210$. This is because Roger needs to go up 6 times and right 4 times. We can write this out as 6 Us for each up and 4 Rs for each right with each possible permutation of this ten letter string being a unique path that Roger can take. We now need to subtract out the ways in which Roger passes through the mousetrap which using the same logic as before, is $\binom{5}{2} \cdot \binom{5}{2} = 10 \cdot 10 = 100$ since there are $\binom{5}{2}$ ways to get from the start to the mousetrap and then another $\binom{5}{2}$ to get from the mousetrap to Saumya. Our final answer is then $210 - 100 = 110$.

13. What is the sum of the integer solutions of the equation $(x^2 + 13x + 21)^{(x^2-6x+8)} = 1$?

Proposed by: Ayush Aggarwal

Answer: $\boxed{4}$

This problem is a classic. Given two integers, $a^b = 1$ when either a is 1, b is 0, or a is -1 and b is even. These three cases are all quadratics, which can be checked individually.

$x^2 + 13x + 21 = 1$ has no integer solutions, so that case is done.

$x^2 - 6x + 8 = 0$ happens when $x = \{2, 4\}$, which are both solutions.

$x^2+13x+21 = -1$ when $x = \{-2, -11\}$. However, when $x = -11$, $x^2 - 6x + 8$ is odd, which isn't a solution. Our only solution from this case is thus -2.

Our final solutions are then $\{-2, 2, 4\}$, which sum to 4.

14. What is $\sqrt[3]{3^3 + 4^3 + 5^3 + 6^3 + \cdots + 22^3}$?

Proposed by: Ayush Aggarwal

Answer: $\boxed{40}$

For this problem we use the fact that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$. Applying this we get that $3^3+4^3+5^3+6^3+\cdots+22^3 = (1+2+3+\cdots+22)^2-(1+2)^2 = (22 \cdot 23/2)^2 - (3)^2 = 64009 - 9 = 64000$. Finally we take the cube root of this which is just 40.

15. An equiangular hexagon has consecutive side lengths of 3, 4, 6, and 6. Then the area of this hexagon can be written as $\frac{a\sqrt{b}}{c}$, where a, b, c are integers and b is not divisible by the square of any prime. Find $a + b + c$.

    Proposed by: Andrew Peng

    Answer: $\boxed{130}$

    Extend the sides of the hexagon to form an equilateral triangle with side length $3+4+6 = 13$. The other sides are $13 - 6 - 6 = 1$ and $13 - 3 - 1 = 9$. Therefore, the area of the hexagon is $\frac{13^2\sqrt{3}}{4} - \frac{3^2\sqrt{3}}{4} - \frac{1^2\sqrt{3}}{4} - \frac{6^2\sqrt{3}}{4} = \frac{123\sqrt{3}}{4}$, so the answer is $123 + 3 + 4 = 130$.