GMC 2022 Team A Solutions

Grand Mega Cool ppl

March 5, 2022

1. What is the sum of the *real* roots of the cubic polynomial $x^3 - x^2 - x - 2$?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{2}$

   The polynomial $x^3 - x^2 - x - 2$ factors as $(x-2)(x^2+x+1)$. x^2+x+1 is a quadratic, and through the quadratic formula, it has only complex roots. Thus, the only real root of $x^3 - x^2 - x - 2$ is 2, which is our answer.

2. A painter can paint a painting in 15 days. He and his apprentice can together do it in 10 days. How long does it take the apprentice to paint a painting alone?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{30}$

   The rate at which the painter paints is $\frac{1}{15}$ paintings per day. Let the rate of his apprentice be a paintings per day. We're given that together, their rate is $\frac{1}{10}$ paintings per day, so we have $\frac{1}{15} + a = \frac{1}{10}$, yielding $a = \frac{1}{30}$ paintings per day. It then takes 30 days for the apprentice to paint a painting.

3. A magic product square is an $N \times N$ grid of squares with each square containing a positive integer such that the product of the numbers in every row, column and main diagonal of the square is the same. A certain 3×3 magic product square has this common product equal to 5832. What integer is in the middle square of the grid?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{18}$

   Consider the following magic square:

   $$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

   Let the common product be P. By the given properties, we have $aei = beh = ceg = P$, and multiplying them together, we have $aibhcge^3 = P^3$. However, we also have that $abc = ghi = P$, so $aibhcge^3 = abcghie^3 = P^2e^3 = P^3$, and so $e = \sqrt[3]{P}$. Since $P = 5832$, the cube root of 5832 is $e = 18$.

4. Let $a\star b = ab+a+b$. If m and n are positive integers such that $0\star m\star 1\star 2\star n\star 3 = 5\star 6\star 7\star 8$, what is the minimum value of $m+n$?

   Proposed by: Roger Fan

   Answer: $\boxed{21}$

   The operation $a \star b$ can be written as $(a+1)(b+1)-1$. By calculation, we even have $a \star b \star c = (a+1)(b+1)(c+1)-1$, and this extends further. Thus, we then have $1 \cdot (m+1) \cdot 2 \cdot 3 \cdot (n+1) \cdot 4 = 6 \cdot 7 \cdot 8 \cdot 9$. Simplifying, we have $(m+1)(n+1) = 126$. $m+1+n+1$ is minimized when $m+1$ and $n+1$ are closest together, which occurs when one is 9 and one is 14. As a result, we have $m+n = 21$.

   Remark. $\langle \mathbb{R}, \star \rangle \cong \langle \mathbb{R}, \cdot \rangle$ *due to the isomorphism* $\phi(x) = x+1$.

5. What is the sum of the integer solutions of the equation $(x^2+13x+21)^{(x^2-6x+8)} = 1$?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{4}$

   This problem is a classic. Given two integers, $a^b = 1$ when either a is 1, b is 0, or a is -1 and b is even. These three cases are all quadratics, which can be checked individually.

   $x^2+13x+21 = 1$ has no integer solutions, so that case is done.

   $x^2-6x+8 = 0$ happens when $x = \{2, 4\}$, which are both solutions.

   $x^2+13x+21 = -1$ when $x = \{-2, -11\}$. However, when $x = -11$, x^2-6x+8 is odd, which isn't a solution. Our only solution from this case is thus -2.

   Our final solutions are then $\{-2, 2, 4\}$, which sum to 4.

6. How many ways are there to tile an 10-by-3 board with 10 indistinguishable 3-by-1 trominos, such that none of the trominos overlap?

   Proposed by Arul Mathur and Alan Lee

   Answer: $\boxed{28}$

   We can employ recursion here. Let $f(n)$ be the number of ways to tile a n-by-3 board with trominos. To find our recursion, consider the last few columns of the board. If we have a vertical tromino that takes a full column, we are left with $f(n-1)$ ways to tile the remaining $n-1$-by-3 board. If the tromino is horizontal, we see that the two rows above it must also have a horizontal tromino, as a vertical tromino cannot fit there. This completes 3 of the columns, and there are $f(n-3)$ ways to tile the remaining part of the board. Our recursion is thus $f(n) = f(n-1) + f(n-3)$. We can quickly check that $f(1) = 1, f(2) = 1$, and $f(3) = 2$. Using these base cases, we find that $f(n) = 28$.

7. What is $\sqrt[3]{3^3+4^3+5^3+6^3+\cdots+22^3}$?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{40}$

   We must use the property that $1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$. Let $S = 3^3+4^3+\cdots+22^3$. By the identity mentioned above, we have $S+8+1 = (1+2+3+$

$\cdots + 22)^2 = 253^2$. Using difference of squares, we get $S = 253^2 - 3^2 = 250 \cdot 256 = 2^9 \cdot 5^3$. Therefore $\sqrt[3]{S} = 40$.

8. Monty Hall runs a game show where there are n closed doors. Behind one randomly chosen door is a car, and behind the other $n-1$ doors are goats. Om, the contestant, is called up to play, and chooses a door to open. However, before the door is opened, Monty Hall opens m of the other $n-1$ doors, revealing only goats. Monty Hall then asks Om whether he would like to switch to one of the other $n-1-m$ unopened doors. Surprisingly, Om astutely notices that switching would exactly double his chances of winning a car. Given that $1 \leq n, m \leq 2022$ and $m < n-1$, how many possible combinations of (n, m) are there?

   Proposed by: Roger Fan

   Answer: $\boxed{1010}$

   The probability of choosing a car on the first turn is $\frac{1}{n}$. If Om does not switch, then this is his probability of winning.

   Otherwise, if he switches, there is a $\frac{n-1}{n}$ probability the car is not in his original door. Assuming that he did not originally choose the car's door, out of the $n-1$ other doors, m are closed, so he has a $\frac{1}{n-m-1}$ chance of switching to the right door. Multiplying these probabilities together yields $\frac{n-1}{n(n-m-1)}$, which we are told is $\frac{2}{n}$.

   Solving, we have $n-1 = 2(n-m-1)$, and $n = 2m+1$. The solutions where $1 \leq n, m \leq 2022$ are then all where n is an odd number greater than 3 but at most 2021, so this gives 1010 solutions.

9. For primes p, there are two solutions to the equation $p \mid (p-5)^{p-5} - (p-6)^{p-6}$, where $a \mid b$ if a divides b. What is the sum of these two solutions?

   Proposed by: Arul Mathur

   Answer: $\boxed{302}$

   We will use modular arithmetic to solve this problem. Considering the equation $\pmod{p}$ gives us the following congruence:

$$(p-5)^{p-5} \equiv (p-6)^{p-6} \pmod{p}.$$

   We simplify the bases $\bmod p$, and we use the fact that $\phi(p) = p-1$ to simplify the exponent. Our relation now becomes

$$(-5)^{-4} \equiv (-6)^{-5} \pmod{p}$$
$$625 \equiv -7776 \pmod{p}$$
$$8401 \equiv 0 \pmod{p}$$

   Therefore $p|8401$. Factoring leaves us with $8401 = 31 \cdot 271$, so the answer is 302.

10. Find $22! \pmod{2024}$.

    Proposed by: Saumya Singhal

    Answer: $\boxed{528}$

    Factoring, we have $2024 = 8 \cdot 11 \cdot 23$. $22!$ is clearly divisible by 8 and 11, and by Wilson's theorem, $22!$ is $22 \pmod{23}$. Let $22! = 88k \pmod{2024}$, and by taking this $\pmod{23}$, we have $88k \equiv 22 \pmod{23}$. $88 \equiv 19 \pmod{23}$, and calculating the inverse of 19 using the Euclidean algorithm, we get 17. Thus, $k \equiv 17 \cdot 22 \equiv 6 \pmod{23}$, so $88k \equiv 528 \pmod{2024}$.

11. Tanush has 2022 distinguishable objects and wants to paint each of them 1 of 6 distinct colors, numbered 1 to 6. However, he requires that the total number of objects painted in the colors 1 and 2 must be odd. Let S be the number of ways there are for him to do this. If k is the largest integer such that 2^k divides S, find k.

    Proposed by: Roger Fan

    Answer: $\boxed{2024}$

    Since we want the total number of objects painted in colors 1 and 2 to be odd, we can group them together as being painted by a single color which we can call color 7 for the moment. We will come back to separating them later, but it is important to group them first. Then for all odd integers n between 1 and 2021 inclusive if we have n total objects painted in color 7, the number of ways this can be done is $\binom{2022}{n} \cdot 4^{2022-n}$. We get $\binom{2022}{n}$ since there are that many ways to choose which n of the 2022 numbers will be painted in color 7 and the 4^{2022-n} comes from the fact that the remaining $2022 - n$ objects can be painted in any of the 4 colors from 3-6.

    Now we need to separate color 7 into colors 1 and 2. This is actually quite simple as continuing from above if we have n total objects with color 7, then there are 2^n ways to paint these n objects in the colors 1 and 2 since each object with color 7 can either be color 1 or color 2.

    Combining these we get for all odd integers n from 1 to 2023, the number of ways is $\binom{2022}{n} \cdot 4^{2022-n} \cdot 2^n$ which simplifies to $\binom{2022}{n} \cdot 2^{4044-n}$. This gets us the sum

    $$\binom{2022}{1} \cdot 2^{4043} + \binom{2022}{3} \cdot 2^{4041} + \cdots \binom{2022}{2019} \cdot 2^{2025} + \binom{2022}{2021} \cdot 2^{2023}.$$

    You may see that this is quite similar to an expression for each term of the expansion of $(2+1)^{2022}$ using the binomial theorem. To get to this we can factor out 2^{2022} from every term and we are left with

    $$2^{2022} \cdot \left(\binom{2022}{1} \cdot 2^{2021} + \binom{2022}{3} \cdot 2^{2019} + \cdots \binom{2022}{2019} \cdot 2^3 + \binom{2022}{2021} \cdot 2^1 \right).$$

    We now want to compute the odd terms of the expansion of $(2+1)^{2022}$ which can be done by using a roots of unity filter. In this case, since we are only doing every second term this is fairly simple and we get that $2^{2022} \cdot \frac{(2+1)^{2022} - (2-1)^{2022}}{2}$. This simplifies to

$2^{2021} \cdot (3^{2022} - 1)$ and since we want the greatest power of 2 that divides this the problem remains to find the greatest power of 2 that divides $3^{2022} - 1$ which we can check using some simple modular arithmetic. We get that this is divisible by 2^3 but not 2^4 so our final answer is $2021 + 3 = 2024$.

12. Richard has a combination lock that has the numbers 1 through 10. It takes in a code of 3 numbers 1-10. However no code can have 2 of the same number consecutively, so there are $10 \cdot 9 \cdot 9 = 810$ total codes. How many of these codes are there such that the sum of its 3 numbers is divisible by 3?

    Proposed by: Roger Fan

    Answer: $\boxed{276}$

    Let the 3 numbers, in order, be a, b and c. If there were no restrictions on a, b, and c, we may use generating functions to find the number of codes where $3|a + b + c$. Our ordinary generating function is $f(x) = (x + x^2 + \cdots + x^{10})^3$, and using the Roots of Unity Filter yields $\frac{f(1)+f(\omega)+f(\omega^2)}{3} = \frac{1000+1+1}{3} = 334$ total such codes ($\omega = e^{\frac{i\pi}{3}}$).

    What if $a = b$? Our generating function with $a = b$ is then $g(x) = (x^2 + x^4 + \cdots + x^{20})(x + x^2 + \cdots + x^{10})$, and taking the roots of unity filter once again yields $\frac{102}{3} = 34$ codes where $3|a + b + c$.

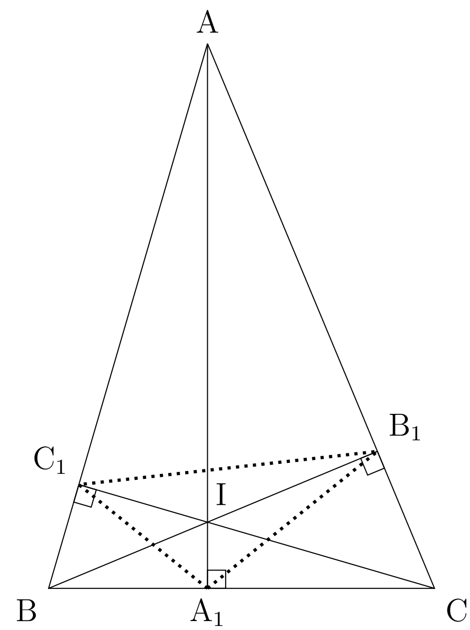
    Finally, if $a = b = c$, we note there are 10 codes for which $3|a + b + c$.

    Taking all these together, we wish to find the number of codes where $a \neq b \neq c$. We now use the principle of inclusion-exclusion: our desired number is just the total number of codes without restrictions, minus the number of codes where $a = b$, minus the number of codes where $b = c$, plus the number of codes where $a = b = c$. The number of codes were $b = c$ is simply the number of codes where $a = b$ by symmetry, so we have $334 - 34 - 34 + 10 = 276$, as claimed.

13. Triangle ABC has $AB = 25$, $BC = 17$, and $AC = 26$. Suppose that from an arbitrary point in the triangle, an infinitely small object is launched so that it bounces infinitely against the walls of the triangle (you may assume it never hits a vertex). Eventually, the motion of this projectile converges to a triangle, which has points A_1, B_1, and C_1 on sides BC, AC, and AB respectively. Let the incenter of triangle $A_1B_1C_1$ be I. If the length of IB can be expressed in the form $\frac{a}{b}$ such that a and b are relatively prime integers, compute $a + b$.

    Proposed by: Steve Zhang

    Answer: $\boxed{103}$



The main idea is that $A_1B_1C_1$ is the orthic triangle of ABC, or that A_1, B_1, and C_1 are the feet of the altitudes from A, B, and C.

We clearly have that $\angle C_1A_1B = \angle B_1A_1C = a$, $\angle A_1B_1C = \angle C_1B_1A = b$, and $\angle B_1C_1A = \angle A_1C_1B = c$. Note that $\angle A + b + c = a + \angle B + c = a + b + \angle C = 180$. From this, it becomes clear that $\angle A = a$, $\angle B = b$, and $\angle C = c$.

It is well known that the incenter of the orthic triangle is simply the orthocenter of the triangle. Thus, I is the orthocenter of ABC. Now construct BB_1 and AA_1. Note that they must intersect at I. Using Heron's we find that $[ABC] = 204$. Since $[ABC] = \frac{BB_1 \cdot AC}{2}$, we get that $BB_1 = \frac{204}{13}$. Similarly, we can find that $AA_1 = 24$, implying that $BA_1 = 7$ by the Pythagorean theorem. Finally, $\frac{IB}{BC} = \frac{A_1B}{BB_1}$ by similar triangles, so $IB = \frac{91}{12}$, resulting in the final answer $91 + 12 = 103$.

14. Equilateral $\triangle ABC$ has center O and side length $12\sqrt{3}$. $\triangle AOB$ is colored red, $\triangle BOC$ is colored blue, and $\triangle COA$ is colored green. A circle with radius 1 is randomly placed such that it's completely contained within $\triangle ABC$. The probability it touches *exactly* 2 colors can be represented as $\frac{a-b\sqrt{3}-\pi}{c\sqrt{3}}$, where a, b, and c are positive integers. Find $a + b + c$.

Proposed by: Roger Fan

Answer: $\boxed{280}$

Consider the center of the circle, M. Note that M must be contained in the equilateral triangle with center O and side length $10\sqrt{3}$.

Let P be the region colored red. Depicted in Figure 1 is the region that M must be in for the circle to touch P. Note that this region consists of P, two trapezoids, and $\frac{1}{6}$ of a circle.

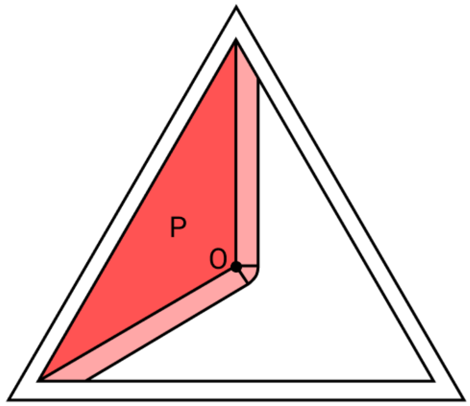


Figure 1: If M is in the pink or red region, the circle will touch the red region.

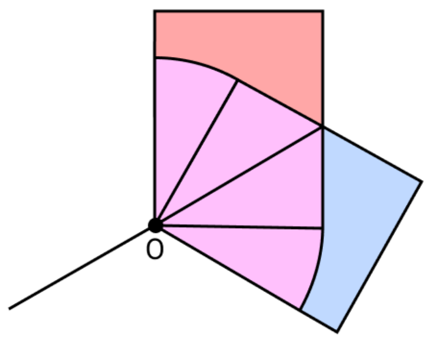


Figure 2: If M lies in the pink region, then the circle touches all 3 colors.

Without loss of generality, assume M lies in the region directly to the right of region P, which is colored green. In Figure 2, we may overlay the region in which M must lie for the circle to touch the red region, with the region in which M must lie for the circle to touch the blue region. The intersection of these two regions, colored in pink, is where M must lie for the circle to touch all 3 circles. Note that it contains both triangles and arcs.

From here, the problem becomes more manageable. Assuming that M lies in the green region, consider the region in which M must lie for the circle to touch *at least* 2 colors. This can be composed of two trapezoids, which together give an area of $20-\frac{4\sqrt{3}}{3}$. Now, we must omit the region in which M must lie for the circle to touch 3 colors. This is the pink region in Figure 2, and has total area $\frac{\sqrt{3}}{3}+\frac{\pi}{6}$. Thus, the region that M must lie in for the circle to touch exactly 2 colors has area $20-\frac{5\sqrt{3}}{3}-\frac{\pi}{6}$.

In the green region, the total area that M can lie in has area $25\sqrt{3}$. Our desired probability becomes

$$\frac{20 - \frac{5\sqrt{3}}{3} - \frac{\pi}{6}}{25\sqrt{3}} = \frac{120 - 10\sqrt{3} - \pi}{150\sqrt{3}}$$

As a result, our answer is $120 + 10 + 150 = 280$.

15. Roger abhors doing his Epsilonmath homework. He starts with 3 questions to do, denoted $c = 3$, and he finishes when $c = 0$. However, he also starts with a spite value of $s = 1$. Given s, the probability of him getting his next question correct is $\frac{2}{s+2}$. If he gets it right, c decreases by 1. If not, his spite s increases by 1 and he hates the world just a little bit more. On average, how many attempts will it take for him to complete the homework?

Proposed by: Roger Fan

Answer: $\boxed{10}$

Solution by: Timothy Herchen

We will actually solve a more general problem than the original question. Observe that if he does c questions and has f failures (incorrect attempts), he makes $c + f$ total attempts. We will analyze the behavior of the number of failures rather than attempts to simplify matters.

Let $f_n(s)$ be the expected number of failures before successfully completing n questions, beginning with spite s. (The original question is $f_3(1) + 2$.) We observe the following recurrence relations:

$$\begin{aligned} f_0(s) &= 0 && (1) \\ f_{n+1}(s) &= \underbrace{\frac{2}{s+2} f_n(s)}_{\text{success}} + \underbrace{\left(1 - \frac{2}{s+2}\right)(f_{n+1}(s+1) + 1)}_{\text{failure}} \\ &= \frac{2}{s+2} f_n(s) + \frac{s}{s+2}(f_{n+1}(s+1) + 1) && (2) \end{aligned}$$

Suppose that $f_n(s) = a_n s + b_n$. (This is not yet known, but it will allow us to confirm or deny the linearity of f_n.) Then $a_n = b_n = 0$. Applying (2) and expanding the left and right sides, we get:

$$\begin{aligned} f_{n+1}(s) &= \frac{2}{s+2} f_n(s) + \frac{s}{s+2}(f_{n+1}(s+1) + 1) \\ a_{n+1}s + b_{n+1} &= \frac{2}{s+2}(a_n s + b_n) + \frac{s}{s+2}(a_{n+1}(s+1) + b_{n+1} + 1) \\ (s+2)(a_{n+1}s + b_{n+1}) &= 2(a_n s + b_n) + s(a_{n+1}(s+1) + b_{n+1} + 1) \\ a_{n+1}s^2 + (2a_{n+1} + b_{n+1})s + 2b_{n+1} &= a_{n+1}s^2 + (2a_n + a_{n+1} + b_{n+1} + 1)s + 2b_n. \end{aligned}$$

Equating coefficients of s, we obtain

$$\begin{cases} a_{n+1} & = a_{n+1} \\ a_{n+1} & = 2a_n + 1 \\ b_{n+1} & = b_n \end{cases}$$

which has a unique solution. In particular, since $a_0 = b_0 = 0$, we have $a_n = 2^n - 1$ and $b_n = 0$. So $f_n(s) = (2^n - 1)s$. The answer to the original problem is $(2^3 - 1)(1) + 3 = 10$.