GMC Team A

1. (20 points) What is the sum of the *real* roots of the cubic polynomial $x^3 - x^2 - x - 2$?

2. (20 points) A painter can paint a painting in 15 days. He and his apprentice can together do it in 10 days. How long does it take the apprentice to paint a painting alone?

3. (25 points) A magic product square is an $N \times N$ grid of squares with each square containing a positive integer such that the product of the numbers in every row, column and main diagonal of the square is the same. A certain 3×3 magic product square has this common product equal to 5832. What integer is in the middle square of the grid?

4. (25 points) Let $a \star b = ab + a + b$. If m and n are positive integers such that $0 \star m \star 1 \star 2 \star n \star 3 = 5 \star 6 \star 7 \star 8$, what is the minimum value of $m + n$?

5. (30 points) What is the sum of the integer solutions of the equation $(x^2 + 13x + 21)^{(x^2 - 6x + 8)} = 1$?

6. (30 points) How many ways are there to tile an 10-by-3 board with 10 indistinguishable 3-by-1 trominos, such that none of the trominos overlap?

7. (35 points) What is $\sqrt[3]{3^3 + 4^3 + 5^3 + 6^3 + \cdots + 22^3}$?

8. (35 points) Monty Hall runs a game show where there are n closed doors. Behind one randomly chosen door is a car,and behind the other $n-1$ doors are goats. Om, the contestant, is called up to play, and chooses a door to open. However, before the door is opened, Monty Hall opens m of the other $n - 1$ doors, revealing only goats. Monty Hall then asks Om whether he would like to switch to one of the other $n - 1 - m$ unopened doors. Surprisingly, Om astutely notices that switching would exactly double his chances of winning a car. Given that $1 \leq n, m \leq 2022$ and $m < n - 1$, how many possible combinations of (n, m) are there?

9. (40 points) For primes p, there are two solutions to the equation $p \mid (p-5)^{p-5} - (p-6)^{p-6}$, where $a \mid b$ if a divides b. What is the sum of these two solutions?

10. (45 points) Find $22! \pmod{2024}$.

11. (50 points) Tanush has 2022 distinguishable objects and wants to paint each of them 1 of 6 distinct colors, numbered 1 to 6. However, he requires that the total number of objects painted in the colors 1 and 2 must be odd. Let S be the number of ways there are for him to do this. If k is the largest integer such that 2^k divides S, find k.

12. (55 points) Richard has a combination lock that has the numbers 1 through 10. It takes in a code of 3 numbers 1-10. However no code can have 2 of the same number consecutively, so there are $10 \cdot 9 \cdot 9 = 810$ total codes. How many of these codes are there such that the sum of its 3 numbers is divisible by 3?

13. (60 points) Triangle ABC has $AB = 25$, $BC = 17$, and $AC = 26$. Suppose that from an arbitrary point in the triangle, an infinitely small object is launched so that it bounces infinitely against the walls of the triangle (you may assume it never hits a vertex). Eventually, the motion of this projectile converges to a triangle, which has points A_1, B_1, and C_1 on sides BC, AC, and AB respectively. Let the incenter of triangle $A_1B_1C_1$ be I. If the length of IB can be expressed in the form $\frac{a}{b}$ such that a and b are relatively prime integers, compute $a + b$.

14. (60 points) Equilateral $\triangle ABC$ has center O and side length $12\sqrt{3}$. $\triangle AOB$ is colored red, $\triangle BOC$ is colored blue, and $\triangle COA$ is colored green. A circle with radius 1 is randomly placed such that it's completely contained within $\triangle ABC$. The probability it touches *exactly* 2 colors can be represented as $\frac{a-b\sqrt{3}-\pi}{c\sqrt{3}}$, where $a, b,$ and c are positive integers. Find $a + b + c$.

15. (70 points) Roger abhors doing his Epsilonmath homework. He starts with 3 questions to do, denoted $c = 3$, and he finishes when $c = 0$. However, he also starts with a spite value of $s = 1$. Given s, the probability of him getting his next question correct is $\frac{2}{s+2}$. If he gets it right, c decreases by 1. If not, his spite s increases by 1 and he hates the world just a little bit more. On average, how many attempts will it take for him to complete the homework?