GMC 2022 Individual B Solutions

Grand Mega Cool ppl

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1. Ayush the astronaut is on a mission to find "Who asked?". To do this, Ayush wants to orbit Earth at a constant height of 100 miles above the equator. However, he realizes there is a satellite in the way so he decides to increase his orbit height to 200 miles. How many additional miles will he travel with this new orbit? Round your answer to the nearest mile.

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{628}$

   It might seem at first that we need the radius of the Earth to solve this problem but we can actually solve it without the exact value. Let's call the radius of the earth R, and we will see that R will cancel out later on.

   The circumference of a circle is $2\pi r$ and so the length of the paths taken by Ayush in his two potential orbits will be $2\pi(R+100)$ and $2\pi(R+200)$. Since we want to know the additional miles he will travel, we take the difference of these two which gives us $2\pi \cdot 100 = 200\pi$. Approximating π as 3.141 we get that $200\pi = 628.2$ which rounds to 628.

2. Arul the ant walks 1 mile on Monday. On each consecutive day he walks 1 more mile than the previous day. How many miles will he have walked by the end of Sunday?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{28}$

   Monday to Sunday is 1 week and so there are 7 days that Arul will walk for. The amount Arul will walk on these 7 days will be 1, 2, 3, 4, 5, 6, and 7 miles. Adding them all up we get 28.

3. Alice's empty swimming pool will hold $30,000$ gallons of water. Two hoses that each supply 5 gallons of water per minute each are used to fill the pool. How many **hours** will it take to fill Alice's pool?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{50}$

   We have 2 hoses that each supply 5 gallons per minute so in total we have $2 \cdot 5 = 10$ gallons per minute. We need 30,000 gallons by the end so we divide 30,000 by 10 to get 3,000 minutes. The problem asks for how many hours it will take so we do $\frac{3{,}000}{60}$ to get our answer of 50 hours.

4. A pouch had 20 candies. Alice took $\frac{2}{5}$ of the candy to eat. Bob took $\frac{2}{3}$ of the remaining candy after Alice took some. Charlie then took all the remaining candies from the pouch. How many candies did Charlie take?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{4}$

   Alice initially takes $20 \cdot \frac{2}{5} = 8$ of the candies. This means that $20 - 8 = 12$ now remain for Bob and Charlie. Bob then takes $12 \cdot \frac{2}{3} = 8$ of the remaining candies leaving Charlie with $12 - 8 = 4$ candies. Since Charlie takes all the remaining candies, Charlie gets 4 candies.

5. Two concentric circles have radii of 3 and 4, respectively. The area of inside the larger circle but outside the smaller circle can be expressed as $k\pi$ for some value of k. What is k^2?

   Proposed by: Andrew Peng

   Answer: $\boxed{49}$

   The area of a circle with radius r is πr^2. To get the area outside one circle but inside the other, we want to subtract the areas of the two circles so we get $\pi \cdot 4^2 - \pi \cdot 3^2 = 16\pi - 9\pi = 7\pi$. This means that $k = 7$ and so $k^2 = 7^2 = 49$.

6. In an alternate universe, the acceptance rate for Gunn High School is 10% and the acceptance rate for Paly High School is 90%. If Sarah applies to both schools, what is the percentage probability that she will be accepted to **only Paly**?

   Proposed by: Alan Lee

   Answer: $\boxed{81}$

   Since we want the probability Sarah is accepted to **only Paly**, that means Sarah is rejected by Gunn and accepted by Paly. The acceptance rate for Gunn is 10% meaning its rejection rate is $100\% - 10\% = 90\%$. Since we are dealing with probabilities we will divide our percentages by 100 to get decimals. The probability of rejection for Gunn is then 0.9 and the probability of acceptance for Paly is 0.9. To get the probability both occur, we multiply these decimals together to get 0.81. Finally we convert this back to a percentage by multiplying by 100 giving us 81%.

7. Let $S = 1! + 3! + 5! + 7! + 9! + \cdots + 99!$. What are the last two digits of S?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{47}$

   This looks daunting at first, but the key is to realize that after a certain point the last two digits of each term will both be 0 which will mean we can disregard them. Any number that is a multiple of 100 will end in 2 0s so we are looking for factorials that are multiples of 100. 100 factorizes into $2^2 \cdot 5^2$ so that means our number must have at least 2 factors of 2 and at least 2 factors of 5. For factorials the smallest number with 2 factors of 5 will be 10 as it will have both 10 and 5 as factors, giving it the necessary two factors of 5. (10! will also have more than 2 factors of 2 as 2, 4, 6, 8, and 10 all provide at least 1).

Alternatively we can find when numbers end in 2 0s by simply listing out each term to get 1, 6, 120, 5040, 362880, 39916800. Since 11! ends in 2 zeroes, and every greater factorial is just 11! multiplied by some number, everything greater than 11! will also end in 2 zeroes.

We now only need to consider the first 5 terms which are 1, 6, 120, 5040, and 362880. Adding up the last two digits of each we get $1+6+20+40+80=147$. We then again take the last two digits of this to get 47.

8. Daniel and Stephen each choose a random integer from 1 to 22 inclusive. The probability that Daniel's number is strictly greater than Stephen's can be expressed in simplest terms as $\frac{m}{n}$. What is $n-m$?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{23}$

   Since both Daniel and Stephen are each randomly choosing a number, both of them have the same probability of picking a larger number than the other person. However, we need to be careful not to simply just say the probability is $\frac{1}{2}$ as we need to consider that they can choose the same number. The probability that this happens is $\frac{1}{22}$ as if we say that Daniel chooses some arbitrary number, then there is only one number that will match that number out of 22 possible number for Stephen to choose from. Since we know the probability that Daniel picks a bigger number is equal to the probability Stephen picks a bigger one and the probability they choose the same number is $\frac{1}{22}$, the probability Daniel's number is bigger is $\frac{1-\frac{1}{22}}{2}=\frac{21}{44}$. Finally $44-21=23$.

9. A dartboard is created by drawing 2022 concentric circles with radii from 1 to 2022. The point values of the 2022 sections created by the circles are proportional to the area of the section, with the center section having point value 1. What is the point value of the outermost section?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{4043}$

   The center section will just be a circle with radius 1, and the outermost section will be a circle with radius 2022 with a circle of radius 2021 cut out of its center. Note that because all the inner sections are contained within the circle of radius 2021, we really don't need to worry about them. Since the point values are proportional to areas, we find the areas using the formula for the area of a circle, πr^2. The inner section then has area π and the outer section has area $(2022^2-2021^2)\pi$. We can calculate this by squaring 2022 and 2021, but it is far faster to use difference of squares to get that $2022^2-2021^2=(2022+2021)(2022-2021)=4043$. We now have the area of the two sections and since the outermost section is 4043 times the area of the innermost, it has 4043 times the point value giving it a value of $4043\cdot 1=4043$.

10. Let $S(n)$ denote the number of factors of an integer n. Let $T(n)$ denote the number of odd factors of an integer. For how many positive integers $n < 1000$ is $S(n) = 7 \cdot T(n)$?

    Proposed by: Ayush Aggarwal

    Answer: $\boxed{8}$

    For each odd factor of an even number n, we can multiply it by 2 to get another factor of n. In fact, if 2^k divides n, we can multiply any odd factor by 2^k to get an even factor. Thus, the ratio of the odd factors to the even factors must be 1 to k, if 2^k is the largest power of 2 that divides n. From this, we know $S(n) = (k+1)T(n)$, and since the condition states that $S(n) = 7T(n)$, we must count the number of integers $n < 1000$ such that $2^6 = 64$ divides n and 2^7 does not. Noting that these integers are an arithmetic sequence $64, 192, 320, \cdots 960$, we count 8 solutions.