GMC 2022 Individual A Solutions

Grand Mega Cool ppl

March 5, 2022

1. Let $S = 1! + 3! + 5! + 7! + 9! + \cdots + 99!$. What are the last two digits of S?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{47}$

   If $n \geq 10$, $n!$ contains at least 2 factors of 5 and 2 factors of 2, so it must be divisible by 100. To find the last two digits of S, we only need to consider the first 5 terms, $1! + 3! + 5! + 7! + 9!$. Computing the last 2 digits of each of these terms yields $S \equiv 1! + 3! + 5! + 7! + 9! \equiv 1 + 6 + 20 + 40 + 80 \equiv 47 \pmod{100}$, so the last 2 digits of S must be 47.

2. Let n be the number of 4-digit numbers which have at least one digit (in base 10) that is a 2, 3, or 5. Find $10n$.

   Proposed by: Roger Fan

   Answer: $\boxed{69420}$

   Consider the 4-digit numbers that contain no 2s, 3s, or 5s. To be a 4-digit number, the first digit must also be nonzero, which leaves 6 choices for the first digit and 7 for the other 3. Thus, there are $6 \cdot 7^3 = 2058$ numbers without any 2s, 3s, and 5s. The total number of 4-digit numbers is 9000, so subtracting, we get that the number of 4-digit numbers containing at least one 2, 3, or 5 is $n = 9000 - 2058 = 6942$. Thus, $10n = 69420$.

3. Let $S(n)$ denote the number of factors of an integer n. Let $T(n)$ denote the number of odd factors of an integer. For how many positive integers $n < 1000$ is $S(n) = 7 \cdot T(n)$?

   Proposed by: Ayush Aggarwal

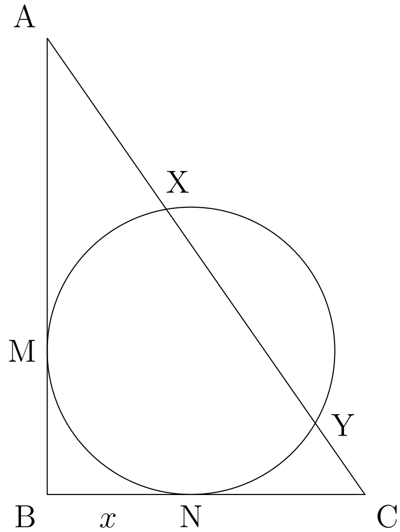
   Answer: $\boxed{8}$

   For each odd factor of an even number n, we can multiply it by 2 to get another factor of n. In fact, if 2^k divides n, we can multiply any odd factor by 2^k to get an even factor. Thus, the ratio of the odd factors to the even factors must be 1 to k, if 2^k is the largest power of 2 that divides n. From this, we know $S(n) = (k+1)T(n)$, and since the condition states that $S(n) = 7T(n)$, we must count the number of integers $n < 1000$ such that $2^6 = 64$ divides n and 2^7 does not. Noting that these integers are an arithmetic sequence $64, 192, 320, \cdots 960$, we count 8 solutions.

4. Right triangle ABC has a right angle at B. Construct a circle such that it is tangent to both AB and BC. AC intersects the circle at points X and Y such that X is on line segment AY. Suppose that $AC = 32$, $AX = 12$, and $CY = 5$. If the radius of the circle is expressed in the form $\sqrt{a} - b$ such that a and b are positive integers, compute $a + b$.

   Proposed by: Steve Zhang

   Answer: $\boxed{510}$



   Suppose that our circle is tangent to AB at point M and BC at point N. Note that by power of a point, we have $AM^2 = AX \cdot AY = 12 \cdot 27 = 324$, implying that $AM = 18$. Similarly, $CN^2 = CY \cdot CX = 5 \cdot 20 = 100$, implying that $CN = 10$. Now note that since BM and BN are both tangents to the circle, we have that $BM = BN$. Let $BM = BN = x$. Furthermore, since $BM \perp BN$, x is also the radius of the circle. Then, by Pythagorean theorem on $\triangle ABC$, we have that

$$(x + 18)^2 + (x + 10)^2 = 32^2.$$

   Solving the quadratic, we get that $x = \sqrt{496} - 14$.

5. Find the sum of all positive k that satisfy the following conditions:

   - $k < 343$
   - $k + 2$ has 3 divisors
   - $k \cdot 2$ has 4 divisors
   - k^2 has 3 divisors

   Proposed by: Alan Lee

   Answer: $\boxed{244}$

We go through the conditions one by one. $k+2$ has 3 divisors if and only if it is the square of a prime. Then, if $2k$ has 4 divisors, counting factors yields that k is either an odd prime or 8. However, $8+2=10$ is not a square, so 8 is not a solution, and k must be an odd prime. Finally, the condition k^2 has 3 factors also tells us that k must be a prime, which we already know.

Thus, we count all integers k such that k is an odd prime and $k+2$ is the square of a prime. The only squares of primes under 343 are $2^2, 3^2, 5^2, 7^2, 11^2, 13^2,$ and 17^2, so setting each one to $k+2$ and checking if k is an odd prime yields $k = \{7, 23, 47, 167\}$. The sum of our solutions is then 244.

6. Let n be the answer to this question. If $x^4 - 8x^3 + 24x^2 - 32x + 14 = n$, find the product of all real values of x.

   Proposed by: Andrew Peng

   Answer: $\boxed{2}$

   $x^2 - 8x^3 + 24x^2 - 32x + 14 = (x-2)^4 - 2 = n$, so we have that the real solutions of x are $x = 2 \pm \sqrt[4]{n+2}$. The product of these two roots is $4 - \sqrt{n-2}$, which we are given is n, so $4 - \sqrt{n+2} = n$. Then, $4 - n = \sqrt{n+2}$ and $n^2 - 8n + 16 = n + 2$. Solving this quadratic yields $n = \{2, 7\}$. However, the solution $n = 7$ is extraneous, as $-3 \neq \sqrt{9}$, so $n = 2$.

7. Sean starts with an initial point (x, y) on the plane. Once every second, he moves the current point (x, y) to the point $(0.1x^2 - 0.1y^2, 0.2xy)$. If we start with some specific initial points, they will get infinitesimally close to the origin as time goes on. Let A be the area of the set of all initial points that approach $(0, 0)$. Find $\lfloor A \rfloor$.

   Proposed by: Roger Fan

   Answer: $\boxed{314}$

   There are many ways to do this problem, but complex numbers provide a natural way of describing the transformation. Consider the function $f(z) = \frac{z^2}{10}$. If $z = x + yi$, then $f(z) = 0.1x^2 - 0.1y^2 + 0.2xyi$. Thus, it suffices to find the region of complex numbers such that after repeated application of $f(z)$, z tends towards 0.

   Consider the magnitude of $f(z)$. $|f(z)| = \frac{|z|}{10}|z|$. If and only if $\frac{|z|}{10} < 1$, then $|f(z)|$ will be smaller than $|z|$, and so $f(z)$ will be closer to the origin than z.

   Thus, all points such that $|z| < 10$ will slowly move towards the origin as time goes on. Meanwhile, all points such that $|z| > 10$ will diverge towards infinity. Our desired set of points is then the region such that $|z| < 10$, which is the circle of radius 10 centered around the origin. The area A is 100π, and $\lfloor A \rfloor = 314$.

   Remark. *This problem is similar in many ways to, surprisingly, the Mandelbrot Set. The Mandelbrot Set is a set of complex numbers that arises from the study of complex dynamics, and is formed by first starting with a point $z = 0$ in the complex plane. Then, given some c, repeatedly move z to the point $z^2 + c$. If z spirals towards infinity, then c is not in the Mandelbrot Set. Otherwise, c* is *in the Mandelbrot Set. The pretty images you see on Google are where each pixel represents a complex number c, and if* *c is in the Mandelbrot Set it's colored black. Otherwise, it's colored some color based on how quickly it diverges. Cool stuff!*

8. A box has 6 slips of paper numbered 1 through 6. 6 people, also conveniently numbered 1 through 6, each draw a slip of paper from the box, but out of sheer luck, none of them choose their own number. Each person has an object, and each second, they give their current object to the person whose number they chose. After 6 seconds, let $\frac{m}{n}$ be expected number of people who have their own object. What is $m+n$?

   Proposed by: Roger Fan

   Answer: $\boxed{299}$

   A derangement is a permutation such that no object stays in the same place. Using the principle of inclusion-exclusion, it can be proven that the number of derangements of size n, denoted D_n, is $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

   The problem really is asking that given a random derangement σ, what is the expected number of fixed points of σ^6?[1]

   Break σ down into cycles.[2] Any cycle of size $2, 3$, or 6 will always become the identity permutation when applied 6 times.[3] Because σ is a derangement, there can't be a cycle of length 5, because then the person not included in the cycle must be a fixed point, which by definition isn't allowed. Thus, the only way for σ^6 to not have a fixed point is for σ to have a cycle of length 4.

   Consider any person P. We can find the probability that after σ^6, P does not end up with their original object, which is the same probability that P is in a cycle of length 4. We must first choose (in order) the 3 other people in the cycle with P, and there are $5 \cdot 4 \cdot 3 = 60$ ways to do this. The other 2 people not in the cycle must then be in a cycle of size 2, since there are no fixed points. Thus, there are 60 ways to have P be in a cycle of size 4, and we know there are 265 ways to have a derangement with 6 people. The probability that P is in a cycle of size 4, then, must be $\frac{60}{265} = \frac{12}{53}$. The probability that P ends up with their original object is 1 minus this fraction, which is $\frac{41}{53}$, and since there are 6 total people, the expected number of people who end up with their object is $6 \cdot \frac{41}{53} = \frac{246}{53}$, and $246 + 53 = 299$.

9. Let $f(1) = 0$. Then, for all $n > 1$, let $f(2n) = f(n)^2$, and let $f(2n+1) = f(n)^2 + 1$. How many integers k from 1 to 2022 inclusive are there such that $f(k)) \geq f(k+1)$?

   Proposed by: Roger Fan

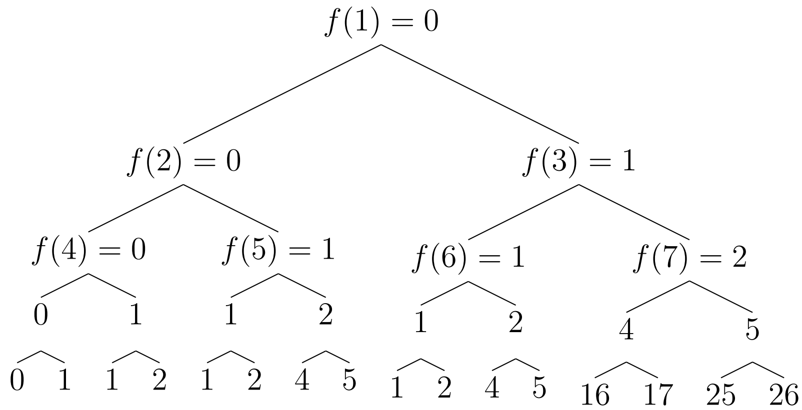
[1]Too much jargon? σ^6 denotes applying σ 6 times, which is what is happening in the problem. A fixed point of a permutation is an object that goes to itself. Thus, a fixed point of σ^6 would be an object that goes back to its owner when σ^6 is applied.

[2]Jargon to English Translation: Any permutation can be broken down into a set of cycles. For example, consider the permutation of size 3 where person 1's object goes to person 2, 2 to 1, and 3 to 3. There's a cycle from 1 to 2 to 1 to 2 and so on, and there's a cycle from 3 to 3 to 3 and so on.

[3]Jargon to English Translation: Consider a cycle where person 1 gives to 2, who gives to 3, and back to 1. When you run this cycle 3 times, then everyone ends up back with their original object, which we call the identity permutation! Convince yourself that every cycle of length m will become the identity permutation after km applications, where k is an integer.

Answer: $\boxed{55}$

It may be helpful to construct a tree where $f(n)$ is connected to $f(2n)$ and $f(2n+1)$, such that each row becomes $f(2^k), f(2^k+1), f(2^k+2), \ldots, f(2^{k+1}-1)$.



We first note that $f(2^n) = 0$ for all integral n, since $f(1) = 0$, and by induction $f(2^n) = f(2^{n-1})^2 = 0$. All other $f(k)$ where k isn't a power of 2 are non-zero.

If $n = 2k$ is even, then we have that $f(n+1) = f(k)^2 + 1 > f(k)^2 = f(2k) = f(n)$, so if $f(n) \geq f(n+1)$, then n must be odd. Let $n = 2k+1$, so that $f(k)^2 + 1 \geq f(k+1)^2$. If $f(k) \geq f(k+1)$, this already is true. However, if $f(k) < f(k+1)$, then $f(k+1)^2 - f(k)^2 \leq 1$ if and only if $f(k+1)$ is 1 or 0 and $f(k)$ is 0, as the difference between $f(k+1)^2$ and $f(k)^2$ would be greater than 1 otherwise.

Thus, we have that $f(2k+1) \geq f(2k+2)$ when $f(k) \leq f(k+1)$ or $f(k) = 0$, $f(k+1) = \{0, 1\}$.

We claim that for all $r > 1$, in the r^{th} row of the tree above, there are $r-1$ values of $f(n)$ where $f(n) \geq f(n+1)$. (In other words, there are $r-1$ values of n where $2^{r-1} \leq n < 2^r$ and $f(n) \geq f(n+1)$.) We prove this by induction.

The base case for $r = 2$ can be computed, for which there indeed is only 1 solution when $f(3) \geq f(4)$.

Assume the hypothesis holds for all previous rows. In the $(r-1)^{\text{th}}$ row, let there be $r-2$ solutions to $f(n) \geq f(n+1)$. Then, from before, we know that $f(2n+1) \geq f(2n+2)$. (Also note $f(2n+1)$ necessarily lies in the r^{th} row.) These form $r-2$ solutions in the r^{th} row, but this is only our first case.

For our second case, $f(2^r+1) = f(2^r+2) = 1$, so that $f(2^r+1) \geq f(2^r+2)$, which provides an extra solution. Because we've covered both cases, our search was exhaustive, and there are exactly $r-1$ solutions in the r^{th} row, and we have proven the hypothesis.[4]

As a final solution, note that in the first row, $f(1) \geq f(2)$, as they are both 0, so there is actually 1 solution in the 1^{st} row, and $r-1$ for the other rows.

[4] There are a couple details missing in this proof, and the reader is welcome to fill in the blanks for themself.

The problem asks to find all k from 1 to 2022 for which $f(k+1) \geq f(k)$. If the problem asked to find all k from 1 to $2047 = 2^{11} - 1$, that would include exactly 11 rows in our tree, giving us $1 + 1 + 2 + \cdots + 10 = 56$ solutions. However, we claim that exactly 1 of these solutions is above 2022, giving us 55 solutions. Namely, this solution is $f(2047) \geq f(2048) = 0$.

It is possible to prove that the second largest solution for which $f(n) \geq f(n+1)$ under 2^n is $3 \cdot 2^{n-2} - 1$ using induction by noting that the second half of each row in our tree always forms a strictly increasing sequence without any 0s. In fact, it is possible to show a stronger statement, that the distance between consecutive solutions in each row are $2, 4, 8, 16, 32, \cdots$ respectively. This result can be proven by induction and its proof would not really add any value here, so it is left as an exercise to the reader.

In the end, however, there are no other solutions under 2048 that are greater than 2022 other than 2047, which leaves us with precisely 55 total solutions.

10. A regular tetrahedron has two vertices at $(0, 0, 0)$ and $(18, 8, 14)$. Let the minimum possible x coordinate of one of the other two vertices be $a - \sqrt{b}$, where a and b are both positive integers. Find $a + b$.

    Proposed by: Ayush Aggarwal

    Answer: $\boxed{204}$

    Because all faces of the tetrahedron are equilateral triangles, it is the same to fix two vertices A and B of an equilateral triangle and find the minimum x-coordinate of the third vertex C. For friendlier computation, let $A = (-p, -q, -r)$ and $B = (p, q, r)$.

    As we rotate the equilateral triangle in 3-D space, the path of C traces out an ellipse. C must lie in the perpendicular bisector plane of the segment AB. Because we've chosen the midpoint of AB to be $(0, 0, 0)$, this plane contains the origin and has the normal vector $\langle p, q, r\rangle$. The equation of the plane is then $px + qy + rz = 0$, given that C is (x, y, z).

    The distance from C to the origin must be the length of the altitude of the equilateral triangle. $AB = 2\sqrt{p^2 + q^2 + r^2}$, so we have the altitude is $\sqrt{3(p^2 + q^2 + r^2)}$. Now, we have $x^2 + y^2 + z^2 = 3(p^2 + q^2 + r^2)$.

    Fix the x-coordinate of C. We have $rz = -(px + qy)$, so

    $$r^2(x^2 + y^2 + z^2) = r^2x^2 + r^2y^2 + (px + qy)^2 = (q^2 + r^2)y^2 + (2pqx)y + (r^2 + p^2)x^2$$

    We also have that $x^2 + y^2 + z^2 = 3(p^2 + q^2 + r^2)$, so

    $$(q^2 + r^2)y^2 + (2pqx)y + (r^2 + p^2)x^2 = 3r^2(p^2 + q^2 + r^2)$$

    This is a quadratic in y, and at the minimum x-coordinate of C, there is only 1 unique solution.[5] Thus, we must choose x so that the discriminant of this quadratic is 0.

[5] Think about this in 3-D space, and convince yourself that this is true.

Calculating the discriminant yields

$$4p^2q^2x^2 - 4(q^2+r^2)(x^2(r^2+p^2) - 3r^2(p^2+q^2+r^2)) = 0$$

Solving for x^2 starts out daunting but turns out to be very elegant.

$$(p^2q^2 - r^4 - r^2q^2 - r^2p^2 - p^2q^2)x^2 = -3r^2(p^2+q^2+r^2)(q^2+r^2)$$

$$r^2(p^2+q^2+r^2)x^2 = 3r^2(p^2+q^2+r^2)(q^2+r^2)$$

$$x = \pm\sqrt{3(q^2+r^2)}$$

This is an astonishingly simple answer, and even more surprising is the fact that x is completely independent of p![6]

We can apply this to the original problem by translating A and B such that their midpoint is on the origin. We then have that the minimum x is $-\sqrt{3(4^2+7^2)} = -\sqrt{195}$. Translating back to the original A and B yields $9-\sqrt{195}$, so our final answer is 204.

It turns out there are actually many other ways to do this problem. The minimum x-coordinate occurs when the equilateral triangle is perpendicular to the yz plane, at which point one can solve using other methods.

[6] We thought about excluding the x-coordinate in the problem statement, but we felt like that might be too evil.