GMC Individual A

1. Let $S = 1! + 3! + 5! + 7! + 9! + \cdots + 99!$. What are the last two digits of S?

2. Let n be the number of 4-digit numbers which have at least one digit (in base 10) that is a 2, 3, or 5. Find $10n$.

3. Let $S(n)$ denote the number of factors of an integer n. Let $T(n)$ denote the number of odd factors of an integer. For how many positive integers $n < 1000$ is $S(n) = 7 \cdot T(n)$?

4. Right triangle ABC has a right angle at B. Construct a circle such that it is tangent to both AB and BC. AC intersects the circle at points X and Y such that X is on line segment AY. Suppose that $AC = 32$, $AX = 12$, and $CY = 5$. If the radius of the circle is expressed in the form $\sqrt{a} - b$ such that a and b are positive integers, compute $a + b$.

5. Find the sum of all positive k that satisfy the following conditions:
   - $k < 343$
   - $k + 2$ has 3 divisors
   - $k \cdot 2$ has 4 divisors
   - k^2 has 3 divisors

6. Let n be the answer to this question. If $x^4 - 8x^3 + 24x^2 - 32x + 14 = n$, find the product of all real values of x.

7. Sean starts with an initial point (x, y) on the plane. Once every second, he moves the current point (x, y) to the point $(0.1x^2 - 0.1y^2, 0.2xy)$. If we start with some specific initial points, they will get infinitesimally close to the origin as time goes on. Let A be the area of the set of all initial points that approach $(0, 0)$. Find $\lfloor A \rfloor$.

8. A box has 6 slips of paper numbered 1 through 6. 6 people, also conveniently numbered 1 through 6, each draw a slip of paper from the box, but out of sheer luck, none of them choose their own number. Each person has an object, and each second, they give their current object to the person whose number they chose. After 6 seconds, let $\frac{m}{n}$ be expected number of people who have their own object. What is $m + n$?

9. Let a sequence $\{f_n\}$ begin with $f_1 = 0$. Then, for all $n > 1$, let $f_{2n} = f_n^2$, and let $f_{2n+1} = f_n^2 + 1$. How many integers k from 1 to 2022 inclusive are there such that $f_k \geq f_{k+1}$?

10. A regular tetrahedron has two vertices at $(0, 0, 0)$ and $(18, 8, 14)$. Let the minimum possible x coordinate of one of the other two vertices be $a - \sqrt{b}$, where a and b are both positive integers. Find $a + b$.