GMC Guts

SET 1 [4 points each]

1. What is the distance between $(0, 3)$ and $(3, 7)$?
2. If $3! \cdot 5! \cdot 7! = n!$, find n.
3. How many integers between 2 and 100 are divisible by at least 4 distinct primes?

SET 2 [5 points each]

4. The height of a projectile as a function of time is modelled to be $h(t) = 100 - t^2$. At $t = 0, 2, 4, 6$, and 8, Sarah writes down $h(t)$ on her piece of paper. Similarly, at $t = 1, 3, 5, 7$, and 9, Sophie writes down $h(t)$ on *her* piece of paper. What is the absolute difference between the sum of the numbers on Sarah's paper and Sophie's paper?
5. Andrew has been playing the hit game BTD6, and he needs to beat round 100, which consists of 1 ZOMG. 1 ZOMG is 4 BFBs, 1 BFB is 4 MOABS, 1 MOAB is 4 Ceramics, 1 Ceramic is 2 Rainbows, 1 Rainbow is 2 Zebras, 1 Zebra is 1 Black and 1 White, 1 Black or 1 White is 2 Pinks, and 1 Pink is 1 Red Bloon. How many Red Bloons are in a ZOMG?
6. An infinite geometric series has a second term of 36 and a fourth term of 4. What is the sum of all the possible sums of this geometric series?

SET 3 [6 points each]

7. Anthoney has an unfair 6-sided dice that lands on 6 half of the time. The other sides have an equal chance of being landed on. If Anthoney rolls two of these die simultaneously, let $\frac{m}{n}$ be the probability that they sum to 11, where m and n are relatively prime positive integers. Find $m + n$.
8. Points A, B, C, and D are drawn on a circle in that order. Chords AC and BD are perpendicular and intersect at Point X creating 4 smaller line segments of lengths $24, 16, 42$, and 28. If the area of the circle is $k\pi$, what is k?
9. Alan, Steve and Tunan play the following game:
   a) First, Alan flips a coin and wins if he flips heads.
   b) If Alan doesn't win, Steve flips a coin and wins if he flips heads.
   c) If neither Alan nor Steve win, Tunan flips a coin and wins if he flips heads.

d) If none of the three win, the process starts over.

Let the probability of Alan winning be a, and let the probability of Tunan winning be t. What is $\frac{a}{t}$?

SET 4 [7 points each]

10. Ayush writes the number 679^{397238} on the board. Every 10 seconds, he writes down the sum of the digits of the number on the board and erases the original number. He stops his mindless task when the number has only one digit. What digit will be on the board?

11. What is the maximum number of bishops that can be placed on an 101×101 chessboard such that none of them attack each other? A bishop "attacks" diagonally in all four directions.

12. If all the integers from 1 to 2022 are written out, how many significant digits will be used? A *significant digit* is a nonzero digit that is not to the left of any other nonzero digit.

SET 5 [8 points each]

13. A hollow ice cream cone has slant height 63 and its base has circumference 42π. An beetle is on the base of the cone and wishes to travel to the diametrically opposite point on the base. What is the length of the shortest path that the beetle can take to reach its destination?

14. Find the number of digits in the base-8 expansion of $8^{2021} \cdot 2021^8$.

15. Let a, b, and c be the roots of $p(x) = 2x^3 + 3x^2 - 4x + 1$. Find

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}$$

SET 6 [10 points each]

16. How many three-digit numbers n satisfy the property that there exist more than $\frac{n}{2}$ positive integers $k \leq n$ with $\gcd(k, n) > 1$?

17. Let $\{a_n\}$ be a sequence of real numbers, defined recursively as follows: $\frac{a_{n+1}}{a_n} = a_n^2 + 3a_n + 3$. If $a_4 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{80}$, what is a_1?

18. The quantity $\sum_{n=0}^{2022} \sin^2(2^n\theta)$, where $\theta = \frac{\pi}{24}$ and calculations are made in radians, can be expressed as $\frac{a-\sqrt{b}-\sqrt{c}-\sqrt{d}}{e}$, where a, b, c, d, e are integers. Find the minimum

possible value of $a+b+c+d+e$.

SET 7 [12 points each]

19. The 24 complex numbers x that satisfy $x^8 = 1$, $x^8 = 4096$ or $x^8 = 390625$ are graphed on the complex plane. How many unordered triplets of these 24 points define the vertices of an isosceles triangle?

20. Let $\varphi = \tan^{-1} x + \sum_{n=0}^{5} \tan^{-1} n$, where x is a randomly chosen real number between 0 and 5. Let P be the probability that the angle φ lies in the first quadrant (i.e. $0 \leq \varphi + 2m\pi \leq \frac{\pi}{2}$ for some integer m). If $P = \frac{m}{n}$ and m, n are relatively prime, find $m+n$.

21. Triangle ABC has $AB = 15$, $AC = 14$, and $BC = 13$. Let its incenter be I, and let its incircle be tangent to AB, AC, and BC at points Z, Y, and X respectively. There are two points, Y' and Z' which lie on rays $\overrightarrow{IY}$ and $\overrightarrow{IZ}$ such that $YY' + ZZ' = 10$. When the area of triangle $XY'Z'$ is maximized, the value of $|YY' - ZZ'|$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime integers. Find $a+b$.

SET 8: ESTIMATION [Max 15 points each]

22. How many ways are there to go from the bottom left to the top right square in an 8×8 chessboard with a queen without ever going left or down? (The path from (1,1) to (2,2) to (8,8) is distinct from the path directly from (1,1) to (8,8); it matters where the queen stops at each move.) If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15 \min\left(\frac{L}{W}, \frac{W}{L}\right)\right\rfloor$ points.

23. A dodecahedron has 12 pentagonal faces and 20 vertices. How many ways are there to color each vertex red, green, blue, or yellow, where two colorings are considered the same if you can rotate one to become the other? If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15\sqrt{\min\left(\frac{L}{W}, \frac{W}{L}\right)}\right\rfloor$ points.

24. Teng Liang has too much time because he managed to reach self-actualization by age 10. One day he finds himself with 5 standard dice, which he rolls together repeatedly. Each trial, he will sum the number of pips (dots) that show up on his dice and record this value in his notebook. How many rolls are expected for the notebook to contain all of the numbers from 5 to 30? If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15\left(\min\left(\frac{L}{W}, \frac{W}{L}\right)\right)^2\right\rfloor$ points.