GMC 2022 Guts Solutions

Grand Mega Cool ppl

March 5, 2022

SET 1

1. What is the distance between $(0, 3)$ and $(3, 7)$?

   Proposed by: Jinwoo Jeong

   Answer: $\boxed{5}$

   Using the distance formula, our answer becomes $\sqrt{(3-0)^2 + (7-3)^2} = 5$.

2. If $3! \cdot 5! \cdot 7! = n!$, find n.

   Proposed by: Andrew Peng

   Answer: $\boxed{10}$

   Factor $3! \cdot 5! = 6 \cdot 120 = 8 \cdot 9 \cdot 10$, so we have $8 \cdot 9 \cdot 10 \cdot 7! = 10! = n!$ so $n = 10$.

3. How many integers between 2 and 100 are divisible by at least 4 distinct primes?

   Proposed by: Roger Fan

   Answer: $\boxed{0}$

   Notice that the 4 smallest primes $2, 3, 5, 7$, which means that the smallest number divisible by 4 distinct primes is $2 \cdot 3 \cdot 5 \cdot 7 = 210 > 100$, so no integers in the specified range satisfy the given condition.

SET 2

4. The height of a projectile as a function of time is modelled to be $h(t) = 100 - t^2$. At $t = 0, 2, 4, 6$, and 8, Sarah writes down $h(t)$ on her piece of paper. Similarly, at $t = 1, 3, 5, 7$, and 9, Sophie writes down $h(t)$ on *her* piece of paper. What is the absolute difference between the sum of the numbers on Sarah's paper and Sophie's paper?

   Proposed by: Roger Fan

   Answer: $\boxed{45}$

   Because the projectile is consistently losing height, the sum of the numbers on Sarah's paper will be greater than that of Sophie's. Thus, we find

$$\begin{aligned}\sum_{n=0}^{4} 100 - (8-2t)^2 - \sum_{n=0}^{4} 100 - (9-2t)^2 &= \sum_{n=0}^{4} (9-2t)^2 - (8-2t)^2 \\ &= \sum_{n=0}^{4} (81 - 64 - 36t + 32t + 4t^2 - 4t^2) \\ &= \sum_{n=0}^{4} 17 - 4t \\ &= 45,\end{aligned}$$

   as desired.

5. Andrew has been playing the hit game BTD6, and he needs to beat round 100, which consists of 1 ZOMG. 1 ZOMG is 4 BFBs, 1 BFB is 4 MOABS, 1 MOAB is 4 Ceramics, 1 Ceramic is 2 Rainbows, 1 Rainbow is 2 Zebras, 1 Zebra is 1 Black and 1 White, 1 Black or 1 White is 2 Pinks, and 1 Pink is 1 Red Bloon. How many Red Bloons are in a ZOMG?

   Proposed by: Andrew Peng

   Answer: $\boxed{1024}$

   The answer is $4 \cdot 4 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 1024$.

6. An infinite geometric series has a second term of 36 and a fourth term of 4. What is the sum of all the possible sums of this geometric series?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{81}$

   Because the second and fourth terms have a ratio of $9:1$, the common ratio squared, r^2, must be $\frac{1}{9}$. That leaves two possible ratios, $\frac{1}{3}$ and $-\frac{1}{3}$. Thus our first term is either 108 or -108. Using the sum of infinite geometric series formula and letting $r' = \frac{1}{3}$, the sum of our possible answers is

$$\frac{a}{1-r'} + \frac{a}{1-(-r')} = \frac{36/r'}{1-1/3} + \frac{-36/r'}{1-(-1/3)} = \frac{108}{2/3} + \frac{-108}{4/3} = 81.$$

SET 3

7. Anthoney has an unfair 6-sided dice that lands on 6 half of the time. The other sides have an equal chance of being landed on. If Anthoney rolls two of these die simultaneously, let $\frac{m}{n}$ be the probability that they sum to 11, where m and n are relatively prime positive integers. Find $m+n$.

   Proposed by: Andrew Peng

   Answer: $\boxed{11}$

   The probabilities of rolling a 1 through 5 is 1/10. To roll an 11, you can only roll a 5 and a 6. The probability is $2 \cdot 1/10 \cdot 1/2 = 1/10$, so the answer is $1 + 10 = 11$.

8. Points A, B, C, and D are drawn on a circle in that order. Chords AC and BD are perpendicular and intersect at Point X creating 4 smaller line segments of lengths $24, 16, 42$, and 28. If the area of the circle is $k\pi$, what is k?

   Proposed by: Ayush Aggarwal

   Answer: $\boxed{845}$

   Consider segments AX, BX, CX, and DX. By Power of a Point, we know that $AX \cdot CX = BX \cdot DX$. WLOG, let $AX = 28$ and $BX = 16$. Now $CX = 24$ and $DX = 42$. Note that the radius of the circle is the same as the circumradius of triangle ABC. The area of the triangle is $\frac{1}{2}bh = 0.5 \cdot 16 \cdot (24 + 28) = 16 \cdot 26 = 416$. We have $AC = 24 + 28 = 52$, $AB = \sqrt{28^2 + 16^2} = 4\sqrt{65}$, and $BC = \sqrt{16^2 + 24^2} = 8\sqrt{13}$. Using the formula $R = \frac{abc}{4K}$, where R is the circumradius, a, b, c are the lengths of the sides, and K is the area of the triangle, we have

   $$R = \frac{52 \cdot 4\sqrt{65} \cdot 8\sqrt{13}}{4 \cdot 416} = 13\sqrt{5}.$$

   Therefore we have $k = R^2 = 845$.

9. Alan, Steve and Tunan play the following game:

   (a) First, Alan flips a coin and wins if he flips heads.
   (b) If Alan doesn't win, Steve flips a coin and wins if he flips heads.
   (c) If neither Alan nor Steve win, Tunan flips a coin and wins if he flips heads.
   (d) If none of the three win, the process starts over.

   Let the probability of Alan winning be a, and let the probability of Tunan winning be t. What is $\frac{a}{t}$?

   Proposed by: Alan Lee

   Answer: $\boxed{4}$

   The probability that Alan wins is $\frac{1}{2}$, plus $\frac{1}{8} \cdot \frac{1}{2}$ if no one wins in the first round, plus $\frac{1}{8^2} \cdot \frac{1}{2}$ if no one wins the second round, and so on. Thus his total probability of victory is

$$\frac{1}{2}\left(1+\frac{1}{8}+\frac{1}{64}+\cdots\right)=\frac{1}{2}\cdot\frac{1}{1-1/8}=\frac{4}{7}.$$

Similarly, Tunan has a $\frac{1}{8}$, plus $\frac{1}{8^2}$, plus $\frac{1}{8^3}$ and so on chance of winning, which simplifies to $\frac{1}{7}$. Thus, the ratio is 4.

Another way to solve this question is to notice that whenever a new round occurs, Alan has a $\frac{1}{2}$ chance of winning and Tunan a $\frac{1}{8}$ chance. Thus, the ratio between these probabilities is the answer, which is also 4.

SET 4

10. Ayush writes the number 679^{397238} on the board. Every 10 seconds, he writes down the sum of the digits of the number on the board and erases the original number. He stops his mindless task when the number has only one digit. What digit will be on the board?

    Proposed by: Arul Mathur

    Answer: $\boxed{7}$

    Note that for any natural number N, the sum of the digits of N, denoted $s(n)$ has the same remainder as N when divided by 9. Furthermore, since $s(n)$ is obviously less than n, we will eventually arrive at a one-digit number with the same remainder when divided by 9 as N. We can use modular arithmetic to speed up the process of finding the remainder when 679^{397238}, using Euler's totient theorem.

    $$679^{397238} = (9(75)+4)^{397238} \equiv 4^{397238} \equiv 4^{(66206\cdot\phi(9))+2} \equiv 4^2 = 16 \pmod 9.$$

    As 16 has a remainder of 7 when divided by 9, the answer is 7.

11. What is the maximum number of bishops that can be placed on an 101×101 chessboard such that none of them attack each other? A bishop "attacks" diagonally in all four directions.

    Proposed by: Andrew Peng

    Answer: $\boxed{200}$

    If we consider the 201 ascending diagonals, at most one bishop can be placed on each of these diagonals. Finally, since there cannot be a bishop on two opposite corners, we subtract 1. Therefore, the answer is $201 - 1 = 200$. (Note: the optimal positioning of the bishops are on all of the first and last rows, with two bishops missing from the four corners).

12. If all the integers from 1 to 2022 are written out, how many significant digits will be used? A *significant digit* is a nonzero digit that is not to the left of any other nonzero digit.

    Proposed by: Ayush Aggarwal

    Answer: $\boxed{2022}$

    Notice that the first digit of each number that is written is significant and nothing else. Thus, there are as many significant digits as numbers written.

    Remark. *The original intention of the problem was to as for the number of significant digits, where a significant digit is either a nonzero digit OR a zero that has at least one nonzero digit to its right. To solve this problem, notice that one first finds the number of digits written and then subtracts the number of insignificant digits.*

The number of digits written is $9 \cdot 1 + 90 \cdot 2 + 900 \cdot 3 + 1023 \cdot 4 = 6981$. From here, subtract 1 for every multiple of 10, another 1 for every multiple of 100, and another 1 for every multiple of 1000, which leaves us with

$$6981 - 202 \cdot 1 - 20 \cdot 1 - 2 \cdot 1 = 6757$$

as our final answer.

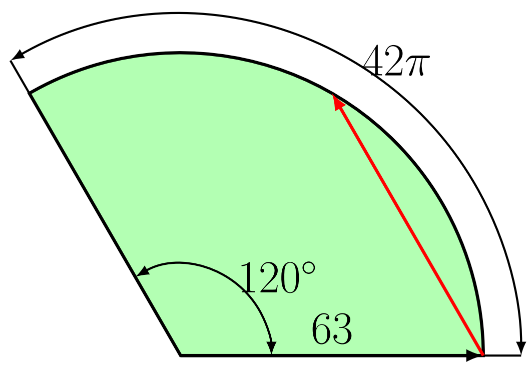
SET 5

13. A hollow ice cream cone has slant height 63 and its base has circumference 42π. An beetle is on the base of the cone and wishes to travel to the diametrically opposite point on the base. What is the length of the shortest path that the beetle can take to reach its destination?

    Proposed by: Ayush Aggarwal

    Answer: $\boxed{63}$

    If we unravel the cone by cutting from its vertex straight down a slant height, we obtain the following net of the cone:



    Notice that in order for the beetle to get to the diametrically opposite point, the beetle must traverse a 60 degree angle on the net. The shortest path between two points on a plane is a line (indicated by the red arrow above). Note that because the red arrow is between two points on the arc 60° apart, it forms an equilateral triangle with the two radii to each point, which means the arrow must also have length 63.

14. Find the number of digits in the base-8 expansion of $8^{2021} \cdot 2021^8$.

    Proposed by: Andrew Peng

    Answer: $\boxed{2051}$

    The number of digits in base-8 of a number n is $\lfloor \log_8 n \rfloor + 1$. Therefore, we need to find $\log_8(8^{2021} \cdot 2021^8)$. Using logarithm rules, we get that it is equal to $2021 + 8\log_8 2021$, which can be approximating knowing that 2021 is slightly less than $2048 = 2^{11}$. Therefore, our approximation is $2021 + 8\log_8 2048 = 2021 + 88/3 = 2050.\overline{3}$. This has a decimal value large enough that we don't have to worry about it being less than 2050, so the final answer is $2050 + 1 = 2051$ digits.

15. Let a, b, and c be the roots of $p(x) = 2x^3 + 3x^2 - 4x + 1$. Find

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}$$

    Proposed by: Arul Mathur

Answer: $\boxed{-9}$

Add $3 = \frac{a}{a} + \frac{b}{b} + \frac{c}{c}$ to the expression to obtain

$$a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

To find $a+b+c$, notice that by Vieta's formulas the sum of the roots is $-\frac{3}{2}$, and for the sum of the reciprocals of the roots, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ac}{abc}=\frac{-4/2}{-1/2}=4.$$

Thus, our product is $-\frac{3}{2}\cdot 4 = -6$. Subtracting the 3 we initially added, we get the final answer of -9.

SET 6

16. How many three-digit numbers n satisfy the property that there exist more than $\frac{n}{2}$ positive integers $k \leq n$ with $\gcd(k, n) > 1$?

Proposed by: Ayush Aggarwal

Answer: $\boxed{458}$

In order for more than half of the numbers under n to share a factor with n, it must have a lot of distinct prime factors. An alternative way to look at the problem statement is if $\phi(n) < \frac{n}{2}$, where ϕ is the totient function. Now we proceed by casework.

- Firstly, no primes work.
- Notice that if $2|n$, as long as it is not a power of 2, $\phi(n)/n < \frac{1}{2}$ because not all of the odd numbers less than n will be relatively prime to n. Thus, if we let $n = 2m$ for $m \neq 2^k$ for integral k, there are $450 - 3 = \mathbf{447}$ choices for m between 50 and 500.
- Now if n is odd, we want to have as many primes divide it as possible. Noting that

$$\frac{\phi(n)}{n} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} > \frac{1}{2},$$

we must have at least 3 distinct prime factors in n. We split this into more cases:

  - $3, 5, 7|n$, which yields that $\phi(n)/n = 2/3 \cdot 4/5 \cdot 6/7 = 48/105 < 1/2$. Thus we can include odd multiples of 105 (105,315,105,735,945), yielding **5** more cases.
  - $3, 5, 11|n$, which yields that $\phi(n)/n = 2/3 \cdot 4/5 \cdot 10/11 = 80/165 < 1/2$. Thus we can include odd multiples of 165 (165,495,825), yielding **3** more cases.
  - $3, 5, 13|n$, which yields that $\phi(n)/n = 2/3 \cdot 4/5 \cdot 12/13 = 96/1195 < 1/2$. Thus we can include odd multiples of 195 (195,585,975), yielding **3** more cases.
  - If $3, 5, p|n$ for $n \geq 17$, $\phi(n)/n > 1/2$ so we need not check these cases.
  - Additionally, if 3 is not a factor of n, one can check that either $n \geq 1000$ or $\phi(n)/n > 1/2$, which means there are no additional solutions for this case.
  - Finally, if more than 3 odd primes divide n, we have $n \geq 3 \cdot 5 \cdot 7 \cdot 11 = 1155$, which is not a 3-digit number.

Adding the valid results together, we obtain $447 + 5 + 3 + 3 = 458$ as our final answer.

17. Let $\{a_n\}$ be a sequence of real numbers, defined recursively as follows: $\frac{a_{n+1}}{a_n} = a_n^2 + 3a_n + 3$. If $a_4 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{80}$, what is a_1?

Proposed by: Roger Fan

Answer: $\boxed{7}$

We can rearrange the given recursion to obtain $a_{n+1} + 1 = (a_n + 1)^3$. Note that $a_4 + 1 = 2^{81}$. Then, $a_3 + 1 = 2^{27}$, $a_2 + 1 = 2^9$, and $a_1 + 1 = 2^3$. Thus, $a_1 = 7$.

18. The quantity $\sum_{n=0}^{2022} \sin^2(2^n\theta)$, where $\theta = \frac{\pi}{24}$ and calculations are made in radians, can be expressed as $\frac{a-\sqrt{b}-\sqrt{c}-\sqrt{d}}{e}$, where a, b, c, d, e are integers. Find the minimum possible value of $a+b+c+d+e$.

Proposed by: Arul Mathur

Answer: $\boxed{12158}$

Notice that if we expand this sum, we obtain

$$\sin^2\left(\frac{\pi}{24}\right)+\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{3}\right)+\sin^2\left(\frac{2\pi}{3}\right)+\sin^2\left(\frac{4\pi}{3}\right)+\sin^2\left(\frac{8\pi}{3}\right)+\cdots.$$

However, note that $8\pi/3 \equiv 2\pi/3 \pmod{2\pi}$, which means that the terms begin to alternate between $\sin^2(2\pi/3) = (\sqrt{3}/2)^2 = 3/4$ and $\sin^2(4\pi/3) = (-\sqrt{3}/2)^2 = 3/4$, leaving us with the new sum

$$\sin^2\left(\frac{\pi}{24}\right)+\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{3}\right)+\sum_{n=4}^{2022}\sin^2(2^n\theta),$$

or

$$\sin^2\left(\frac{\pi}{24}\right)+\sin^2\left(\frac{\pi}{12}\right)+\sin^2\left(\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{3}\right)+\sum_{n=4}^{2022}\frac{3}{4}.$$

Now we find each of the first 4 terms' values. We have

$$\sin^2\left(\frac{\pi}{24}\right) = \frac{1-\cos(\pi/12)}{2} = \frac{1-(\sqrt{6}+\sqrt{2})/4}{2} = \frac{1}{2}-\frac{\sqrt{6}}{8}-\frac{\sqrt{2}}{8},$$

$$\sin^2\left(\frac{\pi}{12}\right) = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \frac{1}{2}-\frac{\sqrt{3}}{4},$$

$$\sin^2\left(\frac{\pi}{6}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4},$$

and

$$\sin^2\left(\frac{\pi}{3}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Now adding our results, we obtain

$$\left(\frac{1}{2}-\frac{\sqrt{6}}{8}-\frac{\sqrt{2}}{8}\right)+\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)+\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)+\sum_{n=4}^{2022}\frac{3}{4},$$

which we can rewrite as

$$\frac{4}{8}-\frac{\sqrt{6}}{8}-\frac{\sqrt{2}}{8}+\frac{4}{8}-\frac{2\sqrt{3}}{8}+\frac{2}{8}+\frac{6}{8}+\frac{12114}{8}=\frac{12130-\sqrt{2}-\sqrt{6}-\sqrt{12}}{8}$$

to obtain a common denominator. Now adding the numbers as instructed, we get the final answer of

$$12130+2+6+12+8=12158.$$

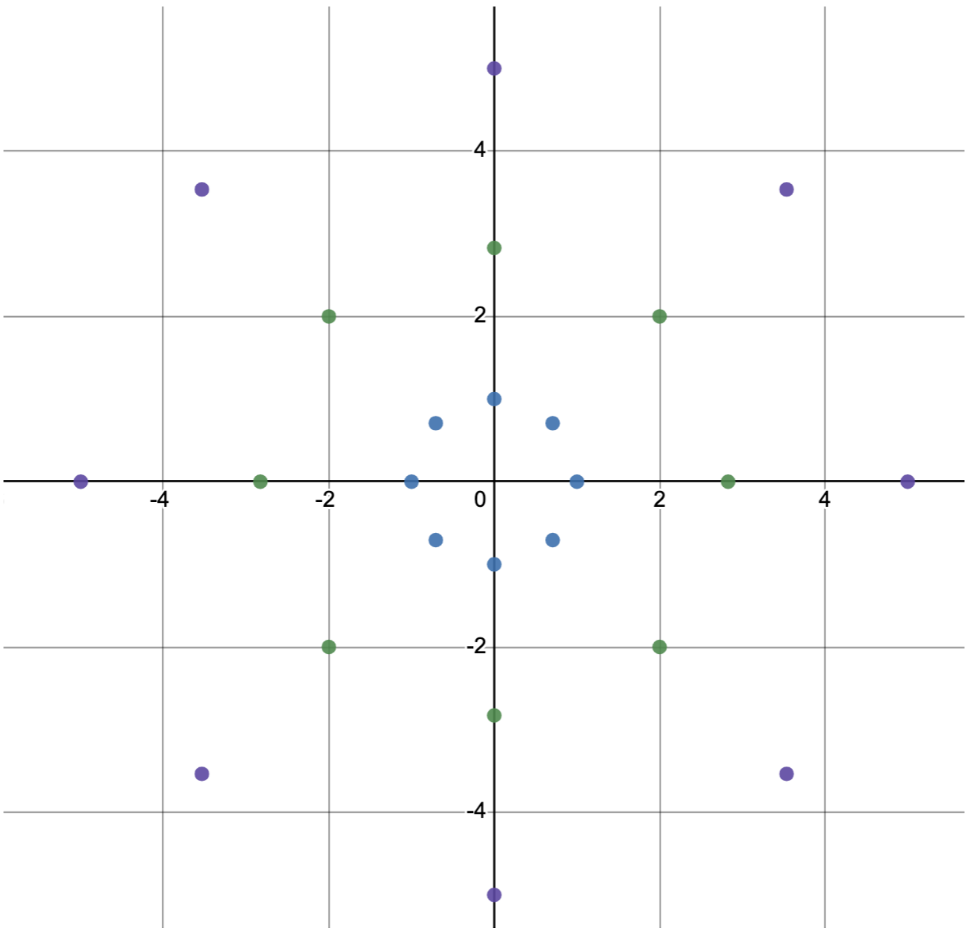
SET 7

19. The 24 complex numbers x that satisfy $x^8 = 1$, $x^8 = 4096$ or $x^8 = 390625$ are graphed on the complex plane. How many unordered triplets of these 24 points define the vertices of an isosceles triangle?

Proposed by: Alan Lee

Answer: $\boxed{248}$

Firstly, because $\sqrt[8]{1} = 1$, $\sqrt[8]{4096} = 2\sqrt{2}$, and $\sqrt[8]{390625} = 5$, the solutions will be of the form $1\text{cis}(\theta), 2\sqrt{2}\text{cis}(\theta), 5\text{cis}(\theta)$ where $\theta \in \{0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{7\pi}{4}\}$. Graphing these points, we obtain the following figure:



Now we can split our isosceles triangles into a few cases:

- If all three vertices of the triangle are the same color, we first have 3 ways to pick the color, 8 ways to pick the center vertex, and 3 ways to pick one of the other vertices in the isosceles triangle. This yields $3 \cdot 8 \cdot 3 =$ **72** total triangles in this case.
- If two vertices share the same color and the last does not, we have 3 ways to pick the color of the center vertex, 8 ways to pick the location of the center vertex, 2 ways to pick the color of the other two vertices, and 3 ways to pick the color of

the other two vertices. This gives us a total of $3 \cdot 8 \cdot 3 \cdot 2 = \mathbf{144}$ total triangles for this case.

- If all three vertices are different colors, we need to do some more in-depth analysis. Firstly, it is important to notice that all such triangles will have legs of length $\sqrt{13}$. Enumerating them one by one, we get
  - Triangles with vertex angle $90°$. For example, we can take $-i$,$2+2i$, and 5. There are 8 triangles of this sort that are oriented clockwise, and another 8 oriented counterclockwise (ie. for each blue point there are 2 possible isosceles triangles). Thus the total number of such triangles is **16**.
  - Triangles with vertex angle θ, where $\theta = 2\arctan(1.5)$. For example, take $-i$, $2+2i$, and $5i$. Once again, for each blue point there are two such triangles that can be made so there are **16** triangles for this case too.

Recalling all four cases, we have a total of $72 + 144 + 16 + 16 = 248$ isosceles triangles.

20. Let $\varphi = \tan^{-1} x + \sum_{n=0}^{5} \tan^{-1} n$, where x is a randomly chosen real number between 0 and 5. Let P be the probability that the angle φ lies in the first quadrant (i.e. $0 \le \varphi + 2m\pi \le \frac{\pi}{2}$ for some integer m). If $P = \frac{m}{n}$ and m, n are relatively prime, find $m+n$.

    Answer: $\boxed{181}$

    Let $\theta = \sum_{n=0}^{5} \tan^{-1} n$. Using complex numbers, we can express θ as the argument of the complex number $z = \prod_{n=0}^{5}(1+ni)$. We may multiply this complex number by any real number factor without changing its argument. Coincidentally, $(1+i)(1+2i)(1+3i) = -10$, so we may scale z such that $z = -(1+4i)(1+5i) = 19 - 9i$.

    z lies in the 3rd quadrant, and $\tan\theta = -\frac{9}{19}$. Working modulo 2π, note $\phi = \tan^{-1} x + \theta = \tan^{-1} x - \tan^{-1}\frac{9}{19}$. To satisfy the conditions, $0 \le \phi \le \frac{\pi}{2}$. ϕ cannot be in the 2nd quadrant, as $\tan^{-1} x$ can be at most $\frac{\pi}{2}$, so we only have the restriction taht $\phi \ge 0$. This implies $\tan^{-1} x \ge \tan^{-1}\frac{9}{19}$, and since $\tan^{-1}$ is an increasing function, $x \ge \frac{9}{19}$. This is a necessary and sufficient condition.

    Out of the reals from 0 to 5, the reals from $\frac{9}{19}$ to 5 satisfy the condition. Our desired probability is thus

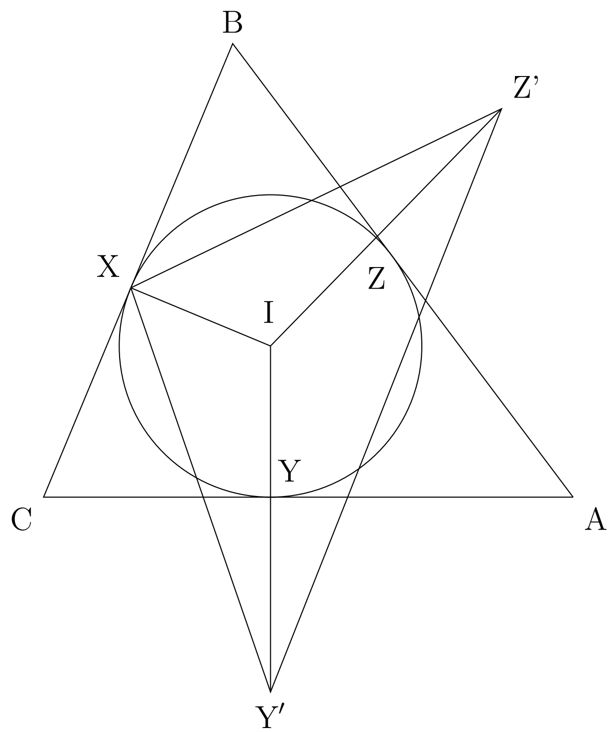
    $$\frac{5 - \frac{9}{19}}{5} = \frac{86}{95}$$

    This gives an answer of $86 + 95 = 181$.

21. Triangle ABC has $AB = 15$, $AC = 14$, and $BC = 13$. Let its incenter be I, and let its incircle be tangent to AB, AC, and BC at points Z, Y, and X respectively. There are two points, Y' and Z' which lie on rays $\overrightarrow{IY}$ and $\overrightarrow{IZ}$ such that $YY' + ZZ' = 10$. When the area of triangle $XY'Z'$ is maximized, the value of $|YY' - ZZ'|$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime integers. Find $a+b$.

    Proposed by: Steve Zhang

    Answer: $\boxed{17}$



We can split the area of triangle $XY'Z'$ into the sum of three separate triangles, $XZ'I$, $XY'I$, and $Y'Z'I$. Now suppose that $YY' = a$. Then, we clearly have $ZZ' = 10 - a$. Furthermore, we can determine that the inradius of ABC is 4, using the fact that the semiperimeter is 21 and the area is 84.

To calculate the areas of each of these triangles, we will use the sine area formula. We know that $\angle Z'IX$ and $\angle B$ are supplementary since $IX \perp BC$ and $IZ \perp AB$. Thus, $\sin(\angle Z'IX) = \sin(\angle B)$. Similarly, $\sin(\angle Y'IX) = \sin(\angle C)$ and $\sin(\angle Y'IZ') = \sin(\angle A)$. Using the sin area formula on triangle ABC, we find that $\sin(\angle A) = \frac{4}{5}$, $\sin(\angle B) = \frac{56}{65}$, and $\sin(\angle C) = \frac{12}{13}$. Thus, we have that

$$[XY'Z'] = [XZ'I] + [XY'I] + [Y'Z'I]$$
$$= \frac{1}{2} \cdot \frac{56}{65} \cdot 4(14 - a) + \frac{1}{2} \cdot \frac{12}{13} \cdot 4(4 + a) + \frac{1}{2} \cdot \frac{4}{5}(4 + a)(14 - a)$$

Since we would like to find the value of a which maximizes this value, we can multiply everything by $\frac{65}{2}$ to obtain

$$56(14 - a) + 60(4 + a) + 13(4 + a)(14 - a) = -13a^2 + 134a + 1752$$

Thus, the maximum occurs at $a = \frac{67}{13}$, implying that $YY' = \frac{67}{13}$, $ZZ' = \frac{63}{13}$, and $|YY' - ZZ'| = \frac{4}{13}$.

SET 8: Estimation

22. How many ways are there to go from the bottom left to the top right square in an 8×8 chessboard with a queen without ever going left or down? (The path from (1,1) to (2,2) to (8,8) is distinct from the path directly from (1,1) to (8,8); it matters where the queen stops at each move.) If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15 \min\left(\frac{L}{W}, \frac{W}{L}\right)\right\rfloor$ points.

    Proposed by: Ayush Aggarwal

    Answer: 1499858

    This answer was calculated via a computer program. The code (in C++) is reproduced below, for reference.

```cpp
#include <iostream>
#include <stdlib.h>
using namespace std;

int main(int argc, const char * argv[]) {

    int finalarray[8][8];
    finalarray[0][0] = 1;
    for (int i=0;i<8;i++){
        for(int j=0;j<8;j++){
            int current = 0;
            for(int k=0;k<j;k++){
                current+=finalarray[i][k];
            }
            for(int l=0;l<i;l++){
                current+=finalarray[l][j];
            }
            if(i>j){
                for(int m=i-1;m-i+j>=0;m--){
                    current+=finalarray[m][m+j-i];
                }
            }else if(i<j){
                for(int m=j-1;m-j+i>=0;m--){
                    current+=finalarray[m+i-j][m];
                }
            }else{
                for(int m=i-1;m>=0;m--){
                    current+=finalarray[m][m];
                }
            }
            finalarray[i][j]=current;
            if(i==j&&i==0){
                finalarray[i][i]=1;
            }
```

```
        }
    }
    for(int i=0;i<8;i++){
        for(int j=0;j<8;j++){
            cout << finalarray[i][j] << " ";
        }
        cout << "\n";
    }
    cout << finalarray[7][7];


}
```

---

23. A dodecahedron has 12 pentagonal faces and 20 vertices. How many ways are there to color each vertex red, green, blue, or yellow, where two colorings are considered the same if you can rotate one to become the other? If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15\sqrt{\min\left(\frac{L}{W}, \frac{W}{L}\right)}\right\rfloor$ points.

    Proposed by: Roger Fan

    Answer: $\boxed{18325477888}$

    The cycle index polynomial of rotations acting on the vertices of a Dodecahedron is

    $$Z = \frac{1}{60}(x_1^{20} + 15x_2^{10} + 20x_1^2x_3^6 + 24x_5^4).$$

    We have 4 colors, so by Burnside's Lemma, we may use $x_i = 4$ for all i. Our desired number of coloring is thus

    $$\frac{1}{60}(4^{20} + 15 \cdot 4^{10} + 20 \cdot 4^8 + 24 \cdot 4^4) = 18325477888.$$

    A good approximation may be made by omitting all terms but the first.

24. Teng Liang has too much time because he managed to reach self-actualization by age 10. One day he finds himself with 5 standard dice, which he rolls together repeatedly. Each trial, he will sum the number of pips (dots) that show up on his dice and record this value in his notebook. How many rolls are expected for the notebook to contain all of the numbers from 5 to 30? If the answer to this problem is W and your answer is L, you will get $\left\lfloor 15\left(\min\left(\frac{L}{W}, \frac{W}{L}\right)\right)^2\right\rfloor$ points.

    Proposed by: Arul Mathur

    Answer: $\boxed{11803}$

    Solution found by: Andrew Peng

    This is an example of the *coupon collector's problem with unequal probabilities*, which one can read more about in Ferrante and Saltalamacchia's paper "The Coupon Collector's Problem", found at `https://mat.uab.cat/matmat_antiga/PDFv2014/v2014n02.pdf`.

In particular, the formula of interest is found on page 7 and for our purposes the answer is

$$\int_0^\infty (1 - ((1 - e^{-1t/6^5})(1 - e^{-5t/6^5})(1 - e^{-15t/6^5})$$
$$(1 - e^{-35t/6^5})(1 - e^{-70t/6^5})(1 - e^{-126t/6^5})$$
$$(1 - e^{-205t/6^5})(1 - e^{-305t/6^5})(1 - e^{-420t/6^5})$$
$$(1 - e^{-540t/6^5})(1 - e^{-651t/6^5})(1 - e^{-735t/6^5})$$
$$(1 - e^{-780t/6^5}))^2)\ \mathrm{d}t = 11803.$$